STOCHASTIC INVESTMENT MODELLING AND PENSION FUNDING: A SIMULATION BASED ANALYSIS

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ABSTRACT

The paper investigates the robustness of theoretical results on the existence of a range of optimal spread periods for a defined benefit pension scheme funded by the projected unit method. These earlier results (see Dufresne (1988), Haberman (1994a), Haberman and Wong (1997)) have been derived using an analytical approach to investigating the behaviour of a simplified model of a defined benefit scheme in the presence of stochastic investment returns. The paper uses a simulation based approach to investigate the distributions of fund level and contribution rate for a defined benefit scheme with a liability structure that includes pensions in payment to be uprated in line with wages inflation, RPI or limited price indexation and an asset structure where investment return are represented by the "Wilkie Model". The effects of including a selection of asset classes, changing from annual to triennial valuations, changing the initial funding level and of using a lower spread period if the fund is in deficit are also investigated. One of the key conclusions is that a range of optimal spread periods continues to exist as these model complexities are introduced.

KEYWORDS: stochastic investment models, pension funding; simulation; Wilkie model.

1 Introduction.

The purpose of this project is to investigate further the theoretical results derived by Dufresne (1988), Haberman(1994a) and Haberman and Wong (1997) on the range of the optimal spread periods for a defined benefit pension scheme funded by the projected unit method. These analytical results are based respectively on the following stochastic models:-

a) annual investment returns being independent and identically distributed (IID)
b) force of interest corresponding to the investment returns following an autoregressive process of order 1 (AR(1))
c) force of interest corresponding to the investment return following a moving average process of order 1 (MA(1)),

and a simplified model of a defined benefit scheme (the structure is described in the appendix).

The use of such simple models is justified on the grounds of mathematical tractability and of providing definitive (usually analytical) results. If the models are constructed on the basis of reasonable, underlying assumptions then there is the possibility that the results will be of relevance to more complex models, which provide a clear representation of the 'real world'. It is unlikely that any such definitive results could be discovered directly from complex and more realistic models, without the prior investigation of simpler models. In particular, it can be difficult to discern conclusively the properties of a system just from the results of simulations. The results from simple models can provide insight but also identify avenues worthy of further investigation based on more complex models (and perhaps with the assistance of simulation). This scenario applies here, with the results of Dufresne (1988), Haberman (1994a) and Haberman and Wong (1997), based on simple models, providing the focus for the follow up investigations presented in this report.

The model of the pension scheme in this project uses the stochastic investment model derived by Wilkie (1995) and the distributions of the fund and the contribution rate are obtained by performing a suitable number of simulations and generating the investment returns for the number of years in each simulation. The model then calculates the fund and contribution rate for each year of each simulation and hence the distribution of the fund and contribution rate are calculated. For each model, a selection of spread periods is tested to investigate the effect that the length of the spread period has on the standard deviation of both the fund and the extra contribution.

In this report, the model of the pension scheme is refined in stages in an attempt to make it as realistic as possible. For the first part of this project (Model I (section 4) - Model VI (section 9)) the theoretical results to be tested are the relationship between the standard deviation of the investment model and the optimal spread period which is described in section 2. Our objective is to see if the results hold as the model becomes more realistic and hence more complex. The later models (Models VII - IX) are used to test other theoretical results derived from the IID investment model.

Model VII changes the pension scheme from annual to triennial valuations and the results are compared with the relevant theoretical results for IID investment returns (see section 10).

Model VIII alters the initial funding level and again the results are compared with the corresponding theoretical results based on IID investment returns (see section 11).

Finally, Model IX (see section 12) uses two different spread periods depending on whether the fund is in surplus or deficit. This enhancement of the model is a natural development, as we seek to move closely to real world practice. No theoretical results are currently available for comparison.

In section 13, three of the models (Models II, IV and VI) are tested to see whether a log-normal approximation can be used to represent the distribution of the fund.

Section 14 presents a summary of the conclusions reached from the results obtained from this project.

2 Background Results.

The main theoretical result to be tested by Models I - VI is the relationship between the range of 'optimal' spread periods and the standard deviation of the investment return. Dufresne (1988) has identified a trade-off between fund security and contribution stability, as measured by the relationship between the limiting variances of the fund level, Var $F(\infty)$, and the contribution rate, Var $C(\infty)$ (as $t \to \infty$), in the presence of IID investment returns. A curve of Var $C(\infty)$ v. Var $F(\infty)$ exhibits a minimum. Then, spread periods greater than the 'optimal' value M^* are unacceptable because there will always be a shorter spread period for which both Var $C(\infty)$ and Var $F(\infty)$ are reduced. In this restricted sense, the desription 'optimal' is used. Further details are provided in section A3 of the Appendix. Table 2.1 shows the value of the optimal spread period for different values of i, the mean return, and σ, the standard deviation of the investment return for the IID model.

Table 2.1 - Value of the Maximum Optimal Spread Period, M^*, for the IID Model as i and σ Vary.

	I				
σ	-0.01	0	0.01	0.03	0.05
0.05	-	401	60	23	14
0.10	-	101	42	20	13
0.15	158	45	28	16	11
0.20	41	26	19	13	10
0.25	22	17	14	10	8

Table 2.1 clearly shows that the maximum optimal spread period, M^*, for the theoretical IID model decreases as either i, and/or σ increases. Therefore, we would expect the spread period for the pension model to change between different versions of the model if these have different values of i or σ or both. The expected effect on the length of the maximum spread period, M^*, will depend on whether i and/or σ increase or decrease as we move from one version of the model to the another.

Section A5.1 of the Appendix introduces the theoretical AR(1) model (Haberman 1994a). Table 2.2 shows the standard deviations of the fund and the extra contribution for this model for a selection of spread periods.

Table 2.2 Relative Standard Deviations of F(t) and C(t) as $t \to \infty$ With i = 0.01, v = 0.05 for the AR(1) Investment Returns Model.

Spread Period	$\frac{(Var\ F(\infty))^{1/2}}{\lvert E\ F(\infty)\rvert}$				$\frac{(Var\ C(\infty))^{1/2}}{\lvert E\ C(\infty)\rvert}$			
	$\varphi = 0.1$	$\varphi = 0.3$	$\varphi = 0.5$	$\varphi = 0.7$	$\varphi = 0.1$	$\varphi = 0.3$	$\varphi = 0.5$	$\varphi = 0.7$
1	5.0%	5.0%	5.0%	5.0%	170%	170%	170%	170%
5	9.1	10.8	12.7	14.2	63.9	77.0	93.8	108
10	12.9	15.7	19.6	25.9	46.5	58.4	77.0	114
20	18.8	23.4	30.3	44.3	35.7	46.5	65.9	121
30	23.9	30.2	40.3	64.6	31.8	42.8	64.7	148
40	28.7	36.7	50.8	94.3	30.2	42.0	68.3	213
50	33.4	43.7	63.1	160	29.6	42.8	76.3	421
60	38.3	51.1	78.5	*	29.8	45.0	90.2	*
80	48.8	69.2	100	*	31.6	53.5	114	*

(* indicates that there is no limiting value).

Table 2.2 shows that a range of optimal spread periods exists for the AR(1) Model as, for each set of parameters, the standard deviation of the fund increases as the spread period increases whereas the standard deviation of the extra contribution decreases at first and then increases as the spread period increases.

Section A5.2 of the Appendix gives a description of the theoretical MA(1) Model (Haberman and Wong 1997). The relative long term standard deviations of the extra contribution and the fund of this model are given in Table 2.3.

Table 2.3 Relative Standard Deviations of F(t) and C(t) as $t \to \infty$ With i = 0.01, v = 0.05 for the MA(1) Investment Returns Model.

Spread Period	$\frac{(Var\ F(\infty))^{1/2}}{\lvert E\ F(\infty)\rvert}$				$\frac{(Var\ C(\infty))^{1/2}}{\lvert E\ C(\infty)\rvert}$			
	φ =-0.3	φ =-0.1	$\varphi = 0.1$	$\varphi = 0.3$	φ =-0.3	φ =-0.1	$\varphi = 0.1$	$\varphi = 0.3$
1	5.0%	5.0%	5.0%	5.0%	157%	157%	157%	157%
5	10.2	9.1	7.7	6.3	66.5	58.7	49.2	39.5
10	14.5	12.8	10.7	8.3	49.1	42.6	34.7	26.5
20	21.3	18.7	15.2	11.6	38.3	32.6	25.9	19.2
30	27.3	23.7	19.1	14.3	34.7	29.1	22.7	16.4
40	33.0	28.4	22.7	16.8	33.4	27.5	21.1	15.0
50	38.8	33.0	26.2	19.2	33.4	27.0	20.2	14.2
60	44.9	37.8	29.7	21.6	34.3	27.1	19.9	13.6
70	51.5	42.8	33.2	23.9	35.9	27.7	19.8	13.2
80	58.8	48.1	36.8	26.2	38.4	28.6	19.9	13.1

Table 2.3 shows that for the MA(1) investment returns model the existance of a range of optimal spread periods is again a feature as for φ = -0.3, φ = -0.01, φ = 0.01 the standard deviation of the fund increases as the spread period increases and the standard deviation of the extra contribution increases and then, beyond a certain point, decreases as the the spread period increases Although this feature is not repeated for φ = 0.03, it can be seen that the standard deviation of the extra contribution is increasing at a decreasing rate as the spread period increases. We would, therefore, conclude that it is likely that an optimal spread period would exist if we continued to increase the value of the spread period.

In section A2 of the Appendix there is also the theoretical result (for the IID Model) that for given values of i and σ there is a spread period of length M_0 such that increasing the spread period above this length will mean that the standard deviation of the fund and the contribution rate will not converge as t tends to infinity. The theoretical results of M_0 are given in Table 2.4.

Table 2.4 - Maximum Length of Spread Period, M_0, for a<1

	i		
σ	0.01	0.03	0.05
0.05	223	111	78
0.10	112	68	51
0.15	66	46	37
0.20	42	33	28
0.25	30	25	21

This theoretical result also holds for autoregressive and moving average models: for numerical examples and for further discussion, readers are referred to Haberman (1994a) and Haberman and Wong (1997).

The results in Table 2.4 are not going to be tested in this project but are included for completeness. The main reasons why they will not be tested here are because the values of the spread periods involved are larger than most pension schemes would use in practice and these long spread periods require simulations over very long periods of time before the long term position (of convergence or divergence) would be established.

3 The Investment Model.

The economic data for this reasearch is generated using the stochastic asset model derived by David Wilkie and published in Wilkie (1995)

In this project, stochastic models are required for different types of investment. These are described below (the models in respect of indexed-linked gilts and conventional gilts are described when these particular asset classes are introduced: see sections 7 and 8). The parameter values stated with the description of each model are those derived by Wilkie (1995) using historic data. For certain parts of this project, these parameter values have been modified: the details including the reasons for any changes are described in later sections where appropriate.

Inflation Model.

The model used for the retail price index where Q(t) is the value of the index at time t is :-

$$Q(t) = Q(t\text{-}1).exp\{I(t)\}$$

Therefore I(t) is the rate of inflation over the year (t-1, t). Wilkie's model for I(t) is:-

$$I(t) = QMU + QA.(I(t\text{-}1) - QMU) + QE(t)$$

where $QE(t) = QSD.QZ(t)$

and it is assumed that $QZ(t) \sim$ iid N(0,1)

The parameter values used are (Wilkie (1995)):-

QMU = 0.0473
QA = 0.5773
QSD = 0.0427

Wage Inflation Model.

Defining J(t) as the rate of wage inflation over the year (t, t-1), then Wilkie's model for J(t) is:-

$$J(t) = WW1.I(t) + WW2.I(t\text{-}1) + WMU + WE(t)$$

where $WE(t) = WSD.WZ(t)$

and $WZ(t) \sim$ iid N(0,1)

The parameter values used are (Wilkie (1995)):-

$WW1 = 0.6021$
$WW2 = 0.2671$
$WMU = 0.0214$
$WSD = 0.0233$

Share Dividend Yield Model

Defining Y(t) as the dividend yield on ordinary shares at time t, then Wilkie's model for Y(t) is:-

$$Y(t) = exp\{YW.I(t) + YN(t)\}$$

where $YN(t) = ln\ YMU + YA.(YN(t-1) - ln\ YMU) + YE(t)$

and $YE(t) = YSD.YZ(t)$

and $YZ(t) \sim$ iid N(0,1)

The parameter values used are (Wilkie (1995)):-

$YW = 1.7940$
$YA = 0.5492$
$YMU = 3.77\%$
$YSD = 0.1552$

Share Dividend Model.

Defining D(t) as the value of the ordinary share dividend index at time t, then Wilkie's model for D(t) is:-

$$D(t) = D(t-1).exp\{DW.DM(t) + DX)I(t) + DMU + DY.YE(t-1) + DB.DE(t-1) + DE(t)\}$$

where $DM(t) = DD.I(t) + (1 - DD).DM(t - 1)$

and $DE(t) = DSD.DZ(t)$

and $DZ(t) \sim$ iid N(0,1)

The parameter values used are (Wilkie (1995)):-

DW = 0.5793
DD = 0.1344
DMU = 0.0157
DY = -0.1761
DB = 0.5733
DSD = 0.0671

Share Price Index and Determining the Value of the Company's Assets.

Defining P(t) to be the price index at time t, then it follows that:-

$$P(t) = D(t)/Y(t)$$

However, we are assuming that the company is going to invest the dividends it receives to buy more shares so we can create an index that represents how the value of the fund is growing. If we define PR(t) to be the value of the index at time t then:-

$$PR(t) = PR(t-1) * \frac{(P(t)+D(t))}{P(t-1)}$$

with PR(0) = 1.

4. Model I - Simple Pension Scheme Model with Wages Indexed Pensions

4.1 Description of Model I.

We begin our analysis with a basic model of a defined pension scheme, making the following simplifying assumptions :-

- The workforce, both active and retired, is a stationary population.
- There is a single age of entry.
- There is a single of retirement.
- There are no benefits paid except for the annual pension.
- Pensions in payment increase at the same rate as wages.
- Entry into the scheme, retirement from the scheme, birthdays and payments of contributions and benefits all occur on the 1st January each year.
- The scheme is an N/60ths final salary pension scheme.
- All the assets of the scheme are invested in equities that perfectly track the equity index.
- The value of the assets is always the market price at any given time.
- The final salary of a member is not increased just prior to retirement i.e. it is the exact salary he/she was earning in the year prior to retirement.
- The assumed rates of interest and wage inflation do not change during the simulation.

Valuations of the pension scheme are carried out at discrete time t (for integer values $t = 0, 1, 2 \ldots.$) and the year t is defined as being from time $t \rightarrow t + 1$.

To calculate the annual contribution rate to the scheme, the expected present value of the pension liability and the expected present value of the active liability the following symbols need to be defined:-

C%	- the annual contribution to the scheme as a percentage of salary.
w	- the expected annual wage inflation.
w_t	- the actual wage inflation for year t.
i	- the expected annual return on the scheme's assets.
i_t	- the actual annual return on the assets for year t.
l_x	- the number of people alive at age x in the scheme.
s_x	- the salary scale at age x.
Sal_t	- the salary of a member aged 25 at time t.

4.1.1 Calculation of the annual contribution rate C.

The contribution is derived by assuming that, on the basis of the assumed rates of interest and wage inflation, the expected present value of each member's contributions is equal to the expected present value of their pension.

The contribution rate for a member entering the scheme at time t is therefore:-

$$Sal_t \frac{C\%}{l_{25}S_{25}}\left[l_{25}S_{25}+\left(\frac{1+w}{1+i}\right)l_{26}S_{26}+....+\left(\frac{1+w}{1+i}\right)^{39}l_{64}S_{64}\right]=$$

$$Sal_t \frac{40}{60}\frac{S_{64}}{l_{25}S_{25}}\frac{(1+w)^{39}}{(1+i)^{40}}\left[l_{65}+l_{66}\left(\frac{1+w}{1+i}\right)+l_{67}\left(\frac{1+w}{1+i}\right)^{2}+....\right]$$

$$=> C\% = \frac{40}{60}S_{64}\frac{\frac{(1+w)^{39}}{(1+i)^{40}}\left[l_{65}+l_{66}\left(\frac{1+w}{1+i}\right)+l_{67}\left(\frac{1+w}{1+i}\right)^{2}+....\right]}{\left[l_{25}S_{25}+\left(\frac{1+w}{1+i}\right)l_{26}S_{26}+....+\left(\frac{1+w}{1+i}\right)^{39}l_{64}S_{64}\right]} \tag{1}$$

It can be seen that Sal_t disappears from the calculation as we are assuming that salary growth during the period of employment does not change with t and hence the contribution rate should not change as it is expressed as a percentage of salary.

4.1.2 Expected Present Value of the Liability of the Pension Population.

For a person retiring at time t the expected present value of their pension is :-

$$PV = Sal_{t-1}\frac{S_{64}}{S_{25}}\frac{40}{60}\left[1+\frac{l_{66}}{l_{65}}\left(\frac{1+w}{1+i}\right)+\frac{l_{67}}{l_{65}}\left(\frac{1+w}{1+i}\right)^{2}+....\right]$$

$$= Sal_{t-1}\frac{S_{64}}{S_{25}}\frac{40}{60}\ddot{a}_{65} \tag{2}$$

where $\ddot{a}_{65}$ is calculated at the rate of interest $= \left(\frac{1+i}{1+w}\right)-1$

Of course, the retired population of the scheme also contains members aged 66, 67, and so on where the people who are now aged 66 retired at time t-1, the people aged 67 retired at time t-2, etc. Under the rules of this first version of the pension scheme, pensions in payment are increased at the same rate as wages. It is, therefore possible to determine the annual payment of each pension in terms of the current salary. As each pension is annually increased at the rate of salary inflation the only difference between the annual pension payments for pensioners of different ages is the rate of wage inflation during their last year of employment. This is because their salary was not increased before the value of their initial pension is calculated. If we define $Pen_{t:x}$ to be the pension paid to a pensioner aged x at time t, then:-

$$Pen_{t:x} = \frac{40}{60}\frac{S_{64}}{S_{25}}\frac{Sal_t}{(1+w_{t+65-x})} \tag{3}$$

Therefore the expected present value of the total liability of the retired population at time t which we shall define as Rl_t is:-

$$
\begin{aligned}
Rl_t &= \frac{S_{64}}{S_{25}}\frac{40}{60}\sum_{x=65}^{\omega}\left(\frac{Sal_t}{(1+w_{t+65-x})}l_x\ddot{a}_x\right) \\
&= \sum_{x=65}^{\omega}\left(Pen_{t:x}l_x\ddot{a}_x\right)
\end{aligned}
\tag{4}
$$

4.1.3 Expected Present Value of the Pension Liability of the Active Population.

For each current employee, the expected present value of their pension liability is the present value of their pension minus the expected present value of their future contributions. The present value of this liability for an employee aged x at time t which we will define as $Al_{x:t}$ is therefore:-

$$
Al_{x:t} = \frac{40}{60}Sal_t\frac{s_{64}}{s_{25}}\frac{l_{65}}{l_x}\frac{(1+w)^{64-x}}{(1+i)^{65-x}}\ddot{a}_{65} - \frac{C\%}{s_{25}l_x}\left(\sum_{y=x}^{64}s_y l_y\left(\frac{1+w}{1+i}\right)^{y-x}\right)
\tag{5}
$$

The total active life liability at time t, which we will define as Al_t, is simply the summation:-

$$
\begin{aligned}
Al_t &= \sum_{x=25}^{64} l_x Al_{x:t} \\
&= \frac{40}{60}\frac{Sal_t}{s_{25}}\sum_{x=25}^{64}\left\{l_{65}\left(\frac{1+w}{1+i}\right)^{65-x}\frac{s_{64}\ddot{a}_{65}}{(1+w)} - C\%\left(\sum_{y=x}^{64}s_y l_y\left(\frac{1+w}{1+i}\right)^{y-x}\right)\right\}
\end{aligned}
\tag{6}
$$

4.1.4 Net Cost.

For the model to work it is also necessary to calculate the net cost due at time t. This is simply the difference between the value of the pension payments due and the contributions to be paid at that time. If we define NC_t to be the net cost due at time t then:-

$$
NC_t = \sum_{x=65}^{\omega} Pen_{t:x}l_x - C\%Sal_t\sum_{x=25}^{64}\frac{s_x}{s_{25}}l_x
\tag{7}
$$

4.2 Simulating the Results.

As stated earlier, the purpose of the model is to investigate the behavior of the contribution rate and the size of the fund of the scheme over time. To explain how the model works it is useful to introduce some further terminology:-

As_t - Value of the assets held by the scheme in year t of the simulation just before contributions and pension payments for that year are made.

Tl_t - The value of the total liabilities of the scheme in year t just before the contributions and pension payments for that year are made. This is thus defined as:- $Tl_t = Al_t + Rl_t$

Ul_t - The unfunded liability of the scheme at time t which is defined as :- $Ul_t = Tl_t - As_t$

Ec_t - The extra contribution paid in year t (see below).

Asl_t - Value of the assets held by the scheme in year t of the simulation just after the contributions and pension payments of the scheme have been made. $Asl_t = As_t - NC_t + Ec_t$

The model assumes that contributions and pension payments are made at the start of each scheme year.

4.2.1 Extra Contribution

The extra contribution occurs because it is highly unlikely that the actual investment returns and wage inflation figures are exactly as those expected and hence an unfunded liability, as defined above, will arise. The value of the extra contribution is determined by the size of the unfunded liability and the spread period being used.

The spread period is defined as the number of years over which the extra contribution should be paid in order that the unfunded liability returns to zero if the rates of wage inflation and interest rates experienced over the period were the same as those assumed. We will assume that the value of the extra contribution increases by wage inflation over the spread period of n years and hence the rate of interest use to derive the initial value of the extra contribution would be equal to (1+i)/(1+w) - 1. Hence we can define Ec_t as :-

$$Ec_t = \frac{Ul_t}{\ddot{a}_{\overline{n|}}} \quad \text{calculated at the rate of interest } \frac{(1+i)}{(1+w)} - 1 \qquad (8)$$

4.2.2 Generation of Results

The model generates the results in a number of different stages.

1. The model first generates the economic data required, in this case the wage inflation and the investment return of the fund (i_t), for each year of the simulation using the Wilkie model previously described, with the input of random numbers where appropriate.
2. Al_0 and Rl_0, the present value of the active and retired life liabilities for the scheme when we start the investigation, are calculated.
3. The assets of the scheme at time 0 are set equal to the total present value of the liabilities of the scheme i.e. $As_0 = Tl_0$
4. Because $As_0 = Tl_0$ the unfunded liability and extra contribution for time 0 are also equal to 0.
5. Nl_0 is calculated and hence Asl_0 can also be calculated.

For the rest of the years of the simulation the results are generated as follows:-

1. $As_t = Asl_{t-1} * (1 + i_{t-1})$
2. Al_t, Rl_t and NC_t are calculated as described in the previous section.
3. $Ul_t = Al_t - As_t$
4. Ec_t is calculated as described above.
5. $Asl_t = As_t - NC_t + Ec_t$

The theroretical results discussed in section 2 and the appendix refer to ultimate values of the moments of the fund and contribution rate. We have tended to run simulations for 149 years as an expedient approximation, taking note that in many practical applications projections would be run for up to a maximum of 40 years only.

A problem with the format of these results is that, because wages tend to increase, we would expect the fund of the scheme and the size of the extra contributions also to increase. As the value of the fund and extra contribution increase, so will the magnitude of the variance if it is left unscaled, in which case we would be unable to identify if the extra contribution were truly becoming more volatile as the simulation progresses. To avoid this problem the extra contribution at time t is expressed as a percentage of the total salary at time t, and the size of the fund at time t is expressed in real terms using year 0 as the base year. Although the required fund in real terms will change slightly, because of the differences in wage inflation in past years affecting the pensions in payment, this should not provide too much variation.

4.3 Results from Model I

The results for Model I were generated with different lengths of spread period. Each set of results was derived from 2000 simulations with each simulation creating 119 or 149 years of results depending on the spread period used.

For each spread period, key statistics were recorded from the distributions of the fund in the different years of the simulation. The data recorded are the mean of the fund for that year, the standard deviation and the 1%, 5%, 10%, 25%, 50%, 75%, 90%, 95% and 99% percentiles along with the inter-quartile range. Similar statistics were recorded for the distribution of the extra contribution. These values are all expressed as percentages of the total salary (i.e. payroll) of the scheme.

Also recorded was the number of times that the fund and the extra contribution go above or below certain levels. For the fund, the lower levels are 80%, 85%, 90% and 95% of the required fund for year 0, i.e. the value of the fund at the beginning of the simulation. So the data recorded show the number of times that the real value of the fund fell below these levels during the 2000 simulations. Similarly, the data recorded for the upper levels which are 105%, 110%, 115% and 120% of the initial fund, record the number of times that the real value of the fund was greater than these levels.

For the extra contributions, the corresponding lower levels chosen are -150%, -100%, -50% and -25% of the regular contribution. So if, for example, the regular contribution rate were 10% of salary then the lower levels for the extra contribution rate will be -15%, -10%, -5% and -2.5% of salary. The upper levels for the contribution rate simply mirror the lower levels i.e. 25%, 50%, 100% and 150% of the regular contribution.

For each spread period, the economic data generated by the Wilkie Model was the same in each case in order to enable a true comparison to be made.

As stated above, the model was run 2000 times for 149 years and this was carried out on a number of different bases. In this report, we consider two bases only. For each the main parameters of the Wilkie model were the ones given in the previous section. For the first run, however, we decided to run the model with the values of each standard deviation parameter halved. This is because preliminary simulations showed that, for the longer spread periods, the model became too volatile in the later years of the simulation. For the second run, the full values of the standard deviation parameters were used.

4.3.1 Results With the Standard Deviation Parameters Halved (Run 1).

The modified parameters values are thus :-

QSD = 0.0213
WSD = 0.0116
YSD = 0.0776
DSD = 0.0335

The model was set-up with the assumed rate of interest $i = 10.87\%$ and the assumed rate of wage inflation $w = 6.25\%$. These figures are the rates derived from the Wilkie Model when there are no random deviations, so that the model is run deterministically. This gives a contribution rate of $C = 6.35\%$.

Certain trends are apparent from the detailed results (not presented here). With a spread period of three years, by the seventh or eighth year of the simulation the values of all the characteristics of the distribution of the fund have either reached or are close to their long term values. As the length of the spread period increases, the year of simulation by which these long term values to be reached increases and when the spread period is 60 years the long term values do not appear to have been attained by the end of the 149 years.

Another feature is that as the length of the spread period increases, the value of the median fund in the last year of the simulations decreases. With the exception of moving from a spread period of 3 years to a spread period of 5 years, this downward trend is also true of the mean value of the fund. However, whereas the median value of the fund in the last year is always less than the starting value of the fund, the mean value of the fund for the last year is actually higher than the starting fund for spread periods less than 40 years.

The results for the variability of the fund show that, as the spread period increases, the long-term standard deviation of the fund also increases and that this long-term value takes longer to be realised. The values of the different percentiles of the fund show that the value of the fund for the lower percentiles decreases and the value for the upper percentiles increases as the spread period increases. However, the shape of the distribution of these fund values appears to change as we consider different spread periods. Tables 4.3.1a and 4.3.1b show the median values in the final year of simulation and the difference between the median and each of the percentiles for the different spread periods. The percentiles are grouped in corresponding pairs for easy comparison.

Table 4.3.1a - The Difference Between the Fund Percentiles and the Median for the Different Spread Periods.

Spread Period	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
3	25.71	1.86	2.27	3.43	4.27
5	25.70	2.20	2.40	3.98	4.89
10	25.50	2.78	3.05	4.88	6.22
15	25.15	3.13	3.82	5.51	7.79
20	24.82	3.60	4.61	6.10	9.42
30	24.17	4.56	5.91	7.46	12.76
40	23.37	5.44	6.72	8.96	16.39
60	21.93	6.37	8.26	10.54	21.06

Table 4.3.1b - The Difference Between the Fund Percentiles and the Median for the Different Spread Periods.

Spread Period	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
3	25.71	4.26	5.62	5.96	8.01
5	25.70	4.87	6.58	6.38	9.45
10	25.50	5.78	8.44	7.44	12.59
15	25.15	6.70	10.48	8.32	16.36
20	24.82	7.54	12.38	9.34	20.49
30	24.17	8.80	17.28	11.03	27.70
40	23.37	10.73	23.66	13.13	41.73
60	21.93	12.43	32.82	15.12	61.93

Tables 4.3.1a and b show that, for each spread period the lower percentile of each pair always has the smaller difference between it and the median than the higher percentile and that this gap increases as the percentiles become more outlying. This gap between the lower and upper percentiles becomes more prominent as the spread period increases. For example, for the 5 year spread period, the difference between the median and the 25% percentile is 2.20 and for the 75% percentile it is 2.40. The upper percentile difference is therefore 9% larger than for the lower percentile. For the 1% and 99% percentiles, the difference for the upper percentile is 48% larger than for the lower percentile. Considering the 40 year spread period, the difference for the 75% percentile is 24% larger than that for the 25% percentile and the difference for the 99% percentile is 318% larger than that for the 1% percentile. The distribution of the fund would therefore appear to become more skewed as the spread period is increased.

Considering the contribution rates, some of the results mirror those for the fund while others show different features. For example, the shorter the spread period, the more quickly the long term values of the distribution of the fund are reached. This, of course, should be expected as it is the size of the fund that determines the extra contribution required. Hence, the median contribution for the larger spread periods has an upward trend as we move further into the simulations because the median fund for these spread periods has a downward trend. Although it is not a strictly one-to-one correspondence, the extra contribution rates for the 1% percentile are generally determined by the level of the funds that determine the 99% percentile statistic of the fund. The magnitudes of these two statistics are thus very similar.

The interesting statistic, however, is the standard deviation of the extra contribution. For spread periods up to and including 20 years, the volatility of the extra contribution decreases as the spread period increases for each year of the simulation. However, this trend does not fully continue when the spread period is 30 years or more. Comparing the standard deviations of the extra contribution for each year recorded when the spread period is 20 years and when the spread period is 30 years, we note from the detailed results (not presented here) that for each year up to and including the 49th year of the simulation the standard deviation of the extra contribution is lower for the 30 year spread period. For the 59th and subsequent years, however, the standard deviation of the extra contribution is greater when the spread period is 30 years. Comparing the spread periods of 30 years and 40 years, we note that the same pattern occurs once again, with the standard deviation of the extra contribution being greater for the 30 year spread period until the 59th year, at which point the standard deviation becomes greater for the larger spread period. In the final comparison between the 40 year and 60 year spread period, the same pattern is repeated, only this time by the 49th year of the simulation the standard deviations are equal. Therefore, increasing the spread period to 30 years or more will, in the long run, make both the extra contribution and the fund more volatile.

Table 4.3.2 highlights this point by showing the standard deviations of the extra contribution in the 119th year of the simulation along with the standard deviation of the fund.

Table 4.3.2 - The Standard Deviations of the Fund and the Extra Contribution.

Spread Period (Years)	Standard Deviation of the Extra Contribution in the Final Year (%)	Standard Deviation of the Fund in the Final Year
3	14.20	3.01
5	10.19	3.46
10	7.16	4.41
15	6.37	5.34
20	6.20	6.31
30	6.63	8.48
40	7.38	11.31
60	8.63	15.61

Table 4.3.2 shows that although the standard deviation of the fund continues to increase as the spread period increases, the standard deviation of the extra contribution falls for the spread periods 3 through to 20 but then increases as the spread period

increases beyond this value. Referring to the theoretical results summarised in section 2, we can see that 20 is the maximum optimal spread period for this model and hence the theoretical result that a range of optimal spread periods exists has been demonstrated.

Currently, our model makes no provision for any limitations on either the size of the fund or the contribution and so our results are likely to infringe both the MFR (Minimum Funding Requirement: Pensions Act 1995) and Inland Revenue restrictions (Finance Act 1986) on the size of scheme surpluses. There are two reasons we have allowed this to happen.

Firstly, our current model has the entire fund invested in equities. This means not only that it is more volatile than a diverse fund investing in stocks such as indexed-linked and conventional gilts but also that the size of both the contribution rate and the starting fund are much smaller.

Secondly, the only benefit paid in this scheme is a pension on retirement. Therefore, it is possible for someone to have made contributions from the age of 25 to the age of 64 but if they then die or retire through ill health no benefit is paid. This of course, has made the contribution rate and the size of the actuarial fund smaller still. As a result, limiting the contribution at this stage of the development of the model has been regarded as being a major hindrance to achieving stability.

If these constraints were included into the model, then their effect would be to lower the standard deviation of the extra contribution and increase the standard deviation of the fund.

4.3.2 Results With the Full Standard Deviation Parameters (Run 2).

Simulated results were also obtained when the standard deviation parameters were restored to their full values. Because of the increased variability, all spread periods have data for 149 years, unlike before, and also extra spread periods of 7 and 25 years have been included.

The results indicate that the main effect of using the full standard deviation parameters is (as one would expect) that the standard deviation of the fund increases compared to the same spread period when using the halved deviation parameters. Also, the lower percentiles (i.e. those below the median) all have lower values for the full deviation parameters compared to the halved deviation parameters and likewise the upper percentiles all have higher values. However, the trends for both the median and the mean have changed.

When we were using the reduced deviation parameters, there was a downward trend for the mean value of the fund in the final years of the simulation as the length of the spread period increased (this was true apart from moving from a spread period of three years to a spread period of five years). For example, at the end of 119 years, the mean fund value when using a three year spread period was 25.96 whereas the value at the end of 119 years when the spread period was increased to sixty years was 25.68. Using the full deviation parameters however, there is an upwards trend in the mean value and it is far more definite i.e. after 149 years the mean value of the fund was 26.63 for the three year spread period and 47.73 when the spread period was 60 years.

The reason for this trend in the mean becomes apparent when we take a closer look at how the values of the percentiles have actually changed. Table 4.3.3 below shows the changes in the values of the percentiles for the spread periods of 3 years and 60 years. For this table, the values of the parameters used to calculate the differences are simply those from the last year of the simulation.

Table 4.3.3 - The Change in the Value of the Fund Percentiles Between the Two Models.

	The Difference in the Value of the Percentile of the Two Models for the Respective Spread Period	
Percentile	3 Year Spread Period	60 Year Spread Period
1%	-4.63	-3.16
5%	-3.73	-3.82
10%	-3.21	-4.23
25%	-1.83	-3.67
50%	+0.30	+1.66
75%	+2.38	+21.06
90%	+5.02	+58.21
95%	+6.19	+108.43
99%	+10.36	+307.71

The changes in the values of the percentiles demonstrate why the mean value of the fund has increased and also why the increase has been far more dramatic for the larger spread periods compared to the smaller spread periods.

Of particular note, though, is that the long term median value for all spread periods is greater with the full standard deviations. The reason for this may be a change in the probability of the fund reaching a critical point i.e. a level where it is impossible to get back towards the desired fund. Table 4.3.3 shows that it is clear that the increase in the probability of having a very large fund is greater than the increase in the probability of having a very small fund. In the long-term, this will lead to a greater increase in the number of very large funds compared to the increase in the number of very small funds and thus the median value will increase.

5 Model II - Price Inflation Linked Pensions.

5.1 Description of Model II.

In this and subsequent sections, we consider improvements to the simple model, Model I, presented in section 4. The first limitation of the simple model that we are going to address is the fact that the pensions in payment have been linked to wage inflation rather than price inflation as is the case in most real schemes. Model II, therefore, has the same assumptions as Model I with the following modifications:-

- Pensions in payment increase at the rate of price inflation.
- The rate of price inflation used in the calculation of the contribution rate and liabilities is kept constant throughout the simulation.

5.1.1 Calculation of the annual contribution rate C%.

The calculation of the annual contribution rate to the new scheme is very similar to before although we now need to define:-

inf - the expected annual inflation rate.
inf_t - the actual inflation rate for year t.

The contribution is once again derived by assuming that the assumed rates of interest and wage inflation are the rates that occur, and that the expected present value of each member's contributions is equal to the expected present value of their pension (see equation (1) in section 4.1).

Allowing for the fact that the member still loses a year of salary growth for the year he/she retires, the formula for the contribution rate for a member entering the scheme at time t becomes:-

$$Sal_t \frac{C\%}{l_{25}S_{25}}\left[l_{25}S_{25}+\left(\frac{1+w}{1+i}\right)l_{26}S_{26}+\ldots+\left(\frac{1+w}{1+i}\right)^{39} l_{64}S_{64}\right]=$$

$$Sal_t \frac{40}{60}\frac{S_{64}}{l_{25}S_{25}}\frac{(1+w)^{39}}{(1+i)^{40}}\left[l_{65}+l_{66}\left(\frac{1+inf}{1+i}\right)+l_{67}\left(\frac{1+inf}{1+i}\right)^2+\ldots\right]$$

$$=> C\% = \frac{40}{60}S_{64}\frac{\frac{(1+w)^{39}}{(1+i)^{40}}\left[l_{65}+l_{66}\left(\frac{1+inf}{1+i}\right)+l_{67}\left(\frac{1+inf}{1+i}\right)^2+\ldots\right]}{\left[l_{25}S_{25}+\left(\frac{1+w}{1+i}\right)l_{26}S_{26}+\ldots+\left(\frac{1+w}{1+i}\right)^{39} l_{64}S_{64}\right]} \qquad (9)$$

It can be seen that Sal_t still disappears from the calculation as we are assuming that salary growth during the period of employment does not change with t and hence the contribution rate should not change as it is expressed as a percentage of salary.

5.1.2 Expected Present Value of the Liability of the Pension Population.

For a person retiring at time t, the present value of their pension is :-

$$PV = Sal_{t-1}\frac{S_{64}}{S_{25}}\frac{40}{60}\left[1+\frac{l_{66}}{l_{65}}\left(\frac{1+inf}{1+i}\right)+\frac{l_{67}}{l_{65}}\left(\frac{1+inf}{1+i}\right)^2+....\right]$$

$$= Sal_{t-1}\frac{S_{64}}{S_{25}}\frac{40}{60}\ddot{a}_{65} \qquad (10)$$

where $\ddot{a}_{65}$ is calculated at the rate of interest $= \left(\frac{1+i}{1+inf}\right)-1$

It should be noted that equation (10) is identical to equation (2) from section 4.1, with only the interest rate used to calculate the perpertuity being different.

We must once again consider the retired population of the scheme which contains members aged 66, 67, etc. where the people who are now aged 66 retired at time t-1, the people aged 67 retired at time t-2, etc. Under the rules of the new scheme, pensions in payment are increased only at the rate of price inflation. It is still possible to determine the annual payment of each pension in terms of the current salary.

For this scheme, the annual pension for the pensioners of different ages is still affected by the rate of wage inflation during their last year of employment as once again they are not covered for this because their salary was not increased before the value of their initial pension is calculated. However, this time the value of the pension also loses value with each year of payment at a rate equal to the difference between wage inflation and price inflation. So keeping the definition that $Pen_{t:x}$ is the pension paid to a pensioner aged x at time t, we obtain the following:-

$$Pen_{t:x} = \frac{40}{60}\frac{S_{64}}{S_{25}}\frac{Sal_t}{(1+w_{t+65-x})}\prod_{y=t+66-x}^{t}\frac{(1+inf_y)}{(1+w_y)} \qquad (11)$$

Having redefined the value of $Pen_{t:x}$, we can define the expected present value of the total liability of the retired population at time t, as before (equation (4) in section 4.1), as Rl_t:-

$$Rl_t = \sum_{x=65}^{\omega}\left(Pen_{t:x}l_x\ddot{a}_x\right) \qquad (12)$$

5.1.3 Expected Present Value of the Pension Liability of the Active Population.

The definition of this expected present value is the same as the previous model, apart from the annuity now being calculated at the rate of interest given above. The expected present value of this liability for an employee aged x at time t, $Al_{x:t}$, is therefore:-

$$Al_{x:t} = \frac{40}{60} Sal_t \frac{s_{64}}{s_{25}} \frac{l_{65}}{l_x} \frac{(1+w)^{64-x}}{(1+i)^{65-x}} \ddot{a}_{65} - \frac{C\%}{s_{25} l_x} \left(\sum_{y=x}^{64} s_y l_y \left(\frac{1+w}{1+i} \right)^{y-x} \right) \tag{13}$$

The total active life liability at time t, Al_t, is simply the summation:-

$$\begin{aligned} Al_t &= \sum_{x=25}^{64} l_x Al_{x:t} \\ &= \frac{40}{60} \frac{Sal_t}{s_{25}} \sum_{x=25}^{64} \left\{ l_{65} \left(\frac{1+w}{1+i} \right)^{65-x} \frac{s_{64} \ddot{a}_{65}}{(1+w)} - C\% \left(\sum_{y=x}^{64} s_y l_y \left(\frac{1+w}{1+i} \right)^{y-x} \right) \right\} \end{aligned} \tag{14}$$

5.1.4 Net Cost.

Once again the definition of this remains the same as the value of the parameters in the equation have changed. Therefore the net cost now due is:-

$$NC_t = \sum_{x=65}^{\omega} Pen_{t:x} l_x - C\% Sal_t \sum_{x=25}^{64} \frac{s_x}{s_{25}} l_x \tag{15}$$

5.1.5 Simulating the Results.

The model works in exactly the same way as the first model with the modified definitions of some of the parameters involved, as noted above.

5.2 Model II Results - Pensions Linked to RPI.

As stated in section 4.3, we are going to use the Wilkie model with the standard deviation values halved so that they are the same as for Model I. Model II was simulated 2000 times for 149 years with the same spread periods as for the second part of Model I.

The parameters chosen were i = 10.87%, w = 6.25% and inf = 4.73%. These results give a contribution rate of C = 5.86% and an initial fund of 22.8782.

As previously, the model was started in equilibrium with all past values of wage and price inflation equal to the assumed rates.

The overall results are very similar in their trends to the results obtained from the first model. For example, the mean value of the fund varies only slightly between the different spread periods although the longer spread periods generally have slightly lower values. However, as for Model I, there is a definite trend with the median value of the fund decreasing as the spread period increases i.e. in the final year of simulation the median value is 22.96 when the spread period is three years falling to 19.32 when the spread period is sixty years.

Similarly, increasing the spread period makes the long term values of the upper percentiles increase and the lower percentiles decrease. The standard deviation also increases.

In the discussion of the results for Model I (section 4.3), it was mentioned that because of the use of a portfolio of equities and because there were no benefits except a pension at normal retirement age, the contribution rate and fund were particularly small and volatile. Now that we have made pensions increase by price rather than wage inflation the contribution rate and initial fund have decreased further. This might lead to the expectation that the new model may be more volatile than the first. This hypothesis is supported by the fact that we are using another parameter, namely price inflation, in the derivation of pension liabilities. However, this hypothesis does not appear to be supported by the detailed results (not shown here): comparing any spread period with its counterpart from the first section of Model I, shows that the new model has a lower standard deviation for both the fund and extra contribution rate.

Of course, the principal explanation for the smaller standard deviation is that the standard fund and contribution rate for this new model are smaller. For example, for a three year spread period, if we assume that the standard deviation of the fund is 3 then dividing by the starting fund for Model I we obtain 0.1166. Assuming the standard deviation of the fund for Model II is 2.65 then dividing by the starting fund we obtain 0.1158. So, if we place some form of scaling on the standard deviation values, the two models appear to have very similar levels of variability.

This feature of the models appearing to have the same relative standard deviation is backed up when the frequency that the fund passes through the defined levels is considered. It appears from these results that the frequency with which the new model passes through the various levels is very similar to the results for the corresponding spread periods with the Model I. However, for the outlying levels of both the fund and the contribution the frequency of Model II appears to be less than that of Model I. It would appear, therefore, that the greater stability of Model II may be due to the fund not reaching 'critical levels' as often as Model I.

This more stable fund may be explained by the fact that we are using the assumed rate of wage inflation to calculate the extra contribution rate even though our liabilities are only rising by price inflation. Therefore, the extra contribution brings the actual fund back towards the correct level of the fund more quickly than before and hence there is less chance that the fund will reach a level which is too high or too low from which to recover. The removal of some of these extreme levels of fund will, therefore, lower the variability of both the fund and the extra contribution rate.

The similarity of the variability of the two models is also confirmed when we examine the relationship between the variability of the fund and the extra contribution rate. As before, the standard deviation of the extra contribution falls at first as the length of the spread period increases. When the spread period exceeds 20 years, however, the long term variability of the extra contribution also increases which is identical to the result in Model I. In fact, the results very closely mirror each other when we consider the years where the standard deviations for the larger spread period become greater than those for the shorter spread period. For Model I, the standard deviation for the 30 year spread period becomes greater than that for the 20 year spread period in the 59th year of simulation. The standard deviations for Model II in the 59th year of simulation are 5.42% for the 30 year spread period and 5.43% for the 20 year spread period so that the two values are effectively equal.

A comparison of the standard deviations for the 30 and 40 year spread periods shows that the results for Model I and Model II are similar with the standard deviation of the extra contribution for the 40 year spread period becoming greater than that for the 30 year spread period by year 59 of the simulation.

Finally, a comparison of results for the 40 and 60 year spread periods shows that for Model I the standard deviations for simulation year 49 are the same, while for Model II the standard deviations are effectively the same with the standard deviation for the 60 year spread period being just 0.01% less.

Table 5.2.1 shows the standard deviations of the extra contribution and of the value of the fund in the final year of the simulation for each spread period. This highlights how the standard deviation first falls as the spread period increases and then rises when the spread period goes above 20 years whereas the standard deviation of the fund increases as the spread period increases. This indicates the existence of a range of optimal spread periods, with a maximum value equal to 20 years (as for Model I, see Table 4.3.2).

Table 5.2.1 - The Standard Deviations of the Extra Contribution and the Fund in the Final Year of the Simulation.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)	Standard Deviation of the Fund in the Final Year
3	12.28	2.61
5	8.97	3.05
7	7.50	3.43
10	6.43	3.96
15	5.74	4.81
20	5.60	5.70
25	5.72	6.65
30	5.98	7.66
40	6.74	9.78
60	8.18	13.38

Table 5.2.2 compares the standard deviation of the extra contribution for the final year of the simulation for Model I, using the halved standard deviation parameters, and Model II.

Table 5.2.2 - The Standard Deviations of the Extra Contribution in the Final Year of the Simulation for Models I and II.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)	
	Model I	Model II
3	14.20	12.28
5	10.19	8.97
10	7.16	6.43
15	6.37	5.74
20	6.20	5.60
30	6.63	5.98
40	7.38	6.74
60	8.63	8.18

As has been noted earlier, a possible reason for the standard deviations for Model II being less than the corresponding values for Model I is that Model II has a smaller standard contribution. Table 5.2.2 highlights the point that the variations in the standard deviation for Model II as the spread period increases follows a very similar pattern to that of Model I.

For completeness, Table 5.2.3 shows the standard deviation of the fund in the final year of the simulation for Models I and II.

Table 5.2.3 - The Standard Deviations of the Fund in the Final Year of the Simulation for Models I and II.

Spread Period (years)	Standard Deviation of the Fund in the Final Year	
	Model I	Model II
3	3.01	2.61
5	3.46	3.05
10	4.41	3.96
15	5.34	4.81
20	6.31	5.70
30	8.48	7.66
40	11.31	9.78
60	15.61	13.38

Again, we can see that, taking into account the differences in scale between the desired funds of the two models, the trend in the standard deviation of the fund, as the spread period increases, is very similar for the two models.

6 Model III - Limited Price Indexing.

6.1 Description of Model III.

We note that the 1995 Pensions Act, contains the requirement that, for pensions accruing after 1st April 1997, occupational pension schemes must provide increases to pensions in payment at the rate of 5% per annum or at the rate of increase of RPI if this is lower.

The previous model, Model II, improved on the first model by linking the increases in the pensions in payment to the Retail Price Index rather than the level of wage inflation. Model III takes this one stage further by using Limited Price Indexing (LPI) for the pensions in payment. With LPI, the annual increase for each pension is still linked with the RPI. However, in years where the inflation rate exceeds 5%, the increase in pensions is limited to 5%.

The way this model works is, therefore, identical to the preceding model apart from the value for inf_t which is now changed to:-

$$inf_t = \text{Min (the inflation rate in year t, 5\%)}$$

As we are assuming an inflation rate for our calculations (of contributions and liabilities) of 4.73%, (i.e. below 5%), this new model has the same contribution rate and standard fund as the previous model. This, of course, allows a straightforward comparison between the new results and the previous results in determining the effects of introducing LPI on the variability and distribution of the fund and extra contribution.

6.2 Model III Results

Using LPI has the effect of limiting the annual increase in the pension liability which would lead to the expectation that the new model should be slightly more stable than the previous one. However, as both wage inflation and the return on equities are linked to price inflation (in the Wilkie Model), there is also an argument that in periods of high inflation there could be an increase in the chance of over funding which would lead to an increase in instability.

The detailed results for Model III (not shown here) indicate that introducing the LPI to the pensions in payment has indeed increased the likelihood of over-funding. This follows from the structure of the Wilkie model which is autoregressive so that, once we have entered a high period of inflation, it is possible for this period of high price inflation, linked to high wage inflation and equity returns, to be sustained for a long period of time. During such periods, the gap between the rate of inflation and the rates of wage inflation and equity returns would be larger than for Model II, which would lead to over-funding.

An analysis of how the fund behaves concentrating on the mean, standard deviation and median value of the fund, shows that the differences between Models II and III appear to be very similar to the effect of increasing the standard deviation parameters from half to full values in Model I (though not to the same extent) as both the mean and standard deviation have increased and the median has remained relatively stable. For Model I, increasing the standard deviation parameters has the effect of making the distribution of the fund for all spread periods retain a similar shape, with the upper percentiles increasing and the lower percentiles decreasing. However, a close examination of the distribution of the funds under Model III shows that this is not the case as the effect of the spread periods depends on the length of the spread period and we note that the shape of the distribution of the fund has changed.

From an examination of the standard deviations in the last year of the simulation, we can see that the standard deviation of the fund size increases as the spread period increases and there is the customary pattern of the standard deviation of the extra contribution firstly decreasing and then increasing as the spread period increases. Table 6.2.1 shows the standard deviations of the extra contribution in the final year of the simulation for the previous model (Model II - no limitation on the price indexing of wages) and the current model, Model III.

Table 6.2.1 - The Standard Deviations of the Extra Contribution and Fund for the Final Year of Simulation for Models II and III.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)		Standard Deviation of the Fund in the Final Year	
	Model II	Model III	Model II	Model III
3	12.28	12.41	2.61	2.62
5	8.97	8.94	3.05	3.03
7	7.50	7.46	3.43	3.40
10	6.43	6.47	3.96	3.97
15	5.74	5.97	4.81	4.99
20	5.60	6.05	5.70	6.15
25	5.72	6.42	6.65	7.45
30	5.98	6.96	7.66	8.90
40	6.74	8.22	9.78	11.92
60	8.18	10.30	13.38	16.83

The way in which the introduction of LPI has affected the distribution of the standard deviations of the fund and extra contribution can be seen by examining Table 6.2.1. For spread periods of 10 years and below, the standard deviation of the extra contribution in the final year of simulation is very similar for both models. However, when the spread period increases to 15 years or above the standard deviation is larger for Model III and the gap between the standard deviations increases as the spread period increases. We, therefore, conclude that Model III is more unstable than Model

II for the higher spread periods but appears to have the same level of stability for shorter spread periods.

The other thing to note from this Table 6.2.1 is that the length of the critical spread period, M^* (identifying the maximum of the range of the optimal spread periods), has been reduced from 20 years to 15 years which we would expect as Model III has a greater variability than Model II (appendix, Section A2).

Tables 6.2.2a and 6.2.2b show how the fund in the final year of the simulation is distributed. For each spread period, the tables give the values of the median and the difference in value between each of the recorded percentiles and the median for the final year of the simulation.

Table 6.2.2a -The Difference Between the Median and the Percentiles of the Fund for Models II and III.

		Difference Between the Median and the Respective Percentile			
Spread Period	Median	25th	75th	10th	90th
3	22.54	1.65	1.89	2.94	3.73
5	22.68	1.82	2.18	3.40	4.35
7	22.91	2.12	2.33	3.91	4.88
10	23.04	2.38	2.84	4.32	5.76
15	23.36	2.90	3.72	5.08	7.19
20	23.92	3.43	4.38	6.03	8.98
25	24.23	3.86	5.35	6.65	10.91
30	24.64	4.31	6.20	7.58	13.31
40	25.37	5.43	7.95	9.20	17.86
60	26.42	7.15	10.53	11.80	24.47

Table 6.2.2b -The Difference Between the Median and the Percentiles of the Fund for Models II and III.

		Difference Between the Median and the Respective Percentile			
Spread Period	Median	5th	95th	1st	99th
3	22.54	3.84	4.90	5.10	6.85
5	22.68	4.32	5.73	5.96	7.90
7	22.91	4.93	6.33	6.75	9.01
10	23.04	5.44	7.57	7.37	11.41
15	23.36	6.26	9.50	8.66	15.98
20	23.92	7.52	12.08	10.18	20.14
25	24.23	8.29	14.89	11.03	24.93
30	24.64	9.42	17.91	12.20	32.94
40	25.37	11.38	24.27	14.17	44.71
60	26.42	13.69	34.29	17.33	63.15

Tables 6.2.2a and 6.2.2.b show two important features. Firstly, the median value of the fund increases as the spread period increases. This is the opposite trend to that reported for the median value in Model II. This reversal in trends is identical to that which occurred in Model I when the standard deviation values were changed from their reduced to their full values (see Sections 4.3.1 and 4.3.2).

Secondly, we see that, as the spread period increases the skewness of the distribution increases in a similar way to before. However, the way in which the distribution has changed when we move from Model II to Model III is more clearly demonstrated if we compare the distribution of the fund in the final years for these models based on three particular spread periods viz. 5, 20 and 40 years.

Tables 6.2.3a and 6.2.3b show the difference for the key fund value distribution characteristics between Model II and Model III in the final year (year 149) of the simulation.

Table 6.2.3a - Difference Between the Fund Distribution Characteristics of Models II and III.

Spread Period	Difference in Value Between Model III and Model II					
	Mean	Standard Deviation	1% Percentile	5% Percentile	10% Percentile	25% Percentile
5	-0.11	-0.02	-0.31	-0.07	0.02	0.00
20	1.72	0.44	0.31	0.92	1.16	1.50
40	5.27	2.14	1.71	2.26	2.88	3.55

Table 6.2.3b - Difference Between the Fund Distribution Characteristics of Models II and III.

Spread Period	Difference in Value Between Model III and Model II				
	Median	75% Percentile	90% Percentile	95% Percentile	99% Percentile
5	-0.14	-0.21	0.07	0.05	-0.20
20	1.74	2.18	2.50	2.88	3.27
40	4.89	6.69	8.49	8.18	14.00

We see from Tables 6.2.3a and 6.2.3b that the change in the distribution of the fund between Models II and III is dependent on the length of the spread period. For the 5 year spread period, the differences in the values are small. Although the statistics for Model II appear to be slightly larger than for Model III, this turns out to be simply a function of the choice of the simulation year to make this comparison. We, therefore conclude, that the introduction of LPI has had little effect on the distribution of the fund when the spread period is 5 years.

When the spread period increases to 20 years, however, there is a marked difference in the distribution of the fund. For all the distribution characteristics, the values for

Model III are larger than the corresponding values for Model II. This is different to the change in the distribution of the fund for Model I when we move from reduced standard deviations to full standard deviations where the median and higher percentiles increase in value but the lower percentiles decrease in value (see Table 4.3.3). It should also be noted that the higher the percentile, the greater the difference between the values for the two models. This would indicate that the distribution of the fund has moved to the right and has become more skewed.

Extending the spread period to 40 years has the effect of magnifying the changes that have occurred for a spread period of 20 years i.e. the values of all the distribution characteristics of the fund have increased and, with the exception of the 90% and 95% percentile, the difference between the values of the fund for the two models has increased (and to a greater extent than for the extension of the spread period from 5 to 20 years). The effect of this on the long-term position of the fund has been to change it from being under-funded to being over-funded. This is demonstrated most effectively by looking at the frequency that the fund passes certain levels as shown below.

<u>**Spread Period of Five Years**</u>

Table 6.2.4a - Frequency that the Extra Contribution in the Final Year Passes Through the Specified Boundaries for Models II and III.

	% of Optimum Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
II	4.45%	11.60%	21.65%	35.25%	35.75%	24.30%	14.35%	7.40%
III	4.70%	11.35%	21.85%	36.55%	33.70%	22.20%	13.55%	8.05%

<u>**Spread Period of Twenty Years**</u>

Table 6.2.4b - Frequency that the Extra Contribution in the Final Year Passes Through the Specified Boundaries for Models II and III.

	% of Optimum Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
II	19.45%	28.30%	38.55%	46.85%	37.50%	29.90%	24.10%	19.00%
III	12.20%	19.30%	25.90%	34.95%	49.50%	41.30%	34.05%	28.50%

Spread Period of Forty Years

Table 6.2.4c - Frequency that the Extra Contribution in the Final Year Passes Through the Specified Boundaries for Models II and III.

	% of Optimum Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
II	36.25%	43.85%	50.45%	54.90%	34.75%	30.65%	26.15%	23.45%
III	17.80%	22.70%	27.30%	35.55%	56.50%	50.85%	46.45%	41.80%

We can see from Tables 6.2.4a, 6.2.4b and 6.2.4c that the results noted earlier appear to be borne out. There is little change for the spread period of five years (once again the differences are due to the particular anomalies of the last year of simulation rather than any underlying trends) whereas there has been a shift from under-funding to over-funding for the case of the spread period of 20 years and an even more significant shift for the 40 years spread period.

7 Model IV - Inclusion of Indexed Linked Gilts.

7.1 Description of Model IV.

So far, the improvements in the models have concentrated solely on the calculation of the liabilities. However, as has been mentioned before, the assumption that the entire pension fund is being invested in equities is certainly unrealistic. In reality, pension funds invest in a mixed portfolio which may include indexed linked gilts (ILGs), conventional gilts, property, etc. The next set of models, therefore, introduces a mixed portfolio.

Indexed linked gilts were introduced by the UK government in 1981 with pension funds as potential investors in mind. They were intended to provide protection against price inflation (high inflation having been a problem throughout the 1970's) as they offer a very stable real return as payments are linked to the retail price index, although there is an 8 month time lag. This class of asset would therefore help the pension fund match its future pension liabilities, as both the pensions in payment and the pension fund increases would be dependent on price inflation. Of course, this protection comes at a price, namely that the expected return is less than from equities and so investing the entire fund in ILGs would rapidly increase the contribution required. Thus, a pension fund with a choice of two investment categories available (equities and ILGs) would split its portfolio between equities and ILGs in order to harness the stability of the ILGs without sacrificing the better return of equities.

Model IV uses the simplest type of split portfolio by having the fund invested in the same proportions each year. For example, if a proportion x ($0<x<1$) of the portfolio is invested in equities and a proportion 1-x is invested in indexed linked gilts then the initial fund after the immediate liabilities have been met will be split between the two asset classes, according to these proportions. At the end of the year, the fund is recombined to give the value which is recorded. At the start of the next year the annual contributions are received, the pensions are paid out and the remaining fund is split between equities and indexed-linked stock according to the above fixed proportions. This process is repeated each year and is the only difference in how Model IV operates compared to Model III. This approach is often described as a re-balancing strategy: Cairns (1995). Other investment strategies have been suggested in the literature (e.g. constant proportion portfolio insurance - see Black and Jones (1987) and Boulier and Kanniganti (1995)) but time does not permit us to investigate their properties. In this discussion, we do not consider 'going short' in an asset and hence we have the restriction that $x>0$.

7.2 Modelling of Index-Linked Stocks.

The modelling of the returns from index-linked stock is carried out by using the Wilkie Model (1995 version) with the standard deviation parameter halved. This model for the real yield, R(t), on index-linked stocks at time t, is as follows:-

$$\ln R(t) = \ln RMU + RA.(\ln R(t-1) - \ln RMU) + RE(t)$$

where $RE(t) = RSD.RZ(t)$

and $RZ(t) \sim \text{i.i.d } N(0,1)$

The parameter values used are:-

RMU = 3.86%
RA = 0.4936
RSD = 0.0365

The total nominal returns at time t are RR(t), defined as:-

$$RR(t) = RR(t-1).\{1/R(t) + 1\}.R(t-1).\{Q(t)/Q(t-1)\}$$

7.3 Results.

Model IV was tested using the spread periods of 5, 7, 10, 15, 20, 25, 30 and 40 years. The outlying spread periods of 3 and 60 years were omitted in order to save time as results from earlier models, and preliminary testing of Model IV, showed that the trends displayed by these extreme spread periods could be identified from an examination of the other spread periods.

Model IV was tested for two different portfolios: the first with 85% of the portfolio invested in equities and 15% in ILGs and the second with 70% of the portfolio invested in equities and 30% in ILGs. As Model III corresponds to Model IV with a portfolio of 100% equities, in reality there are three portfolios to compare.

The assumed rates of price inflation, wage inflation and the return on equities are as before and the return on ILGs is 8.89%. The assumed rate of interest, i, for any particular portfolio with a proportion x invested in equities is therefore:-

$$i\% = x * 10.87\% + (1 - x) * 8.89\%$$

As the assumed rate of interest for the different portfolios is not the same, neither is the derived contribution and fund. Comparisons of the standard deviations and the distribution of the fund must therefore be done with care, as the size of the standard deviation will depend somewhat on the size of the fund. This problem has already been encountered when Model I and Model II were compared (see section 5).

The expected outcome of introducing ILGs into the portfolio is to reduce the variability of both the fund and the extra contribution and the results indicate that this is indeed what has occurred. Table 7.3.1 shows the standard deviation of the extra contribution and the fund in the final year of the simulation for the different portfolios and spread periods.

Table 7.3.1 - The Standard Deviations of the Extra Contribution and Fund for the Final Year of the Simulation for Each Spread Period and Portfolio.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)			Standard Deviation of the Fund in the Final Year		
	100% Eqs	85% Eqs	70% Eqs	100% Eqs	85% Eqs	70% Eqs
5	8.94	7.77	6.66	3.03	2.65	2.30
7	7.46	6.54	5.60	3.40	3.01	2.61
10	6.47	5.71	4.84	3.97	3.55	3.05
15	5.97	5.25	4.37	4.99	4.48	3.79
20	6.05	5.26	4.31	6.15	5.47	4.59
25	6.42	5.48	4.45	7.45	6.55	5.46
30	6.96	5.83	4.70	8.90	7.70	6.41
40	8.22	6.68	5.37	11.92	10.08	8.42

It can be seen from Table 7.3.1 that the introduction of ILGs into the portfolio has reduced the standard deviations of both the fund and the extra contribution for any given spread period. Also, the greater is the proportion of ILGs, the greater is the reduction in the standard deviations. As the results above have not been scaled to take into account the trend in the mean, which increases as the proportion of ILGs in the model increases, the reduction in standard deviations has been slightly understated.

The results obtained from this model are in direct support of the theoretical results (section 2) as the critical spread period, M^*, has increased as the variance of the model has decreased. This is shown by the fact that for the 100% equities portfolio there is a definite increase in the standard deviation of the extra contribution (5.97 to 6.05) as the spread period changes from 15 years to 20 years. When the portfolio is changed to 85% equities and hence has a lower variance, there is still an increase in the standard deviation of the extra contribution between the two spread periods but this time it is only an increase of 0.01 rather than the previous increase of 0.08. Finally, when the proportion of ILGs is increased again so that the portfolio is 70% equities and 30% ILGs, the standard deviation of the extra contribution first increases when the spread period changes from 20 years to 25 years.

In section 6.2, it was observed that the implementation of a new model may change not only the mean and standard deviation of the fund and extra contribution but also the shape of the distribution for certain spread periods. Similar analysis is needed this time but is made more difficult by the fact that the fund and standard contribution are different for the different portfolios.

To see the effect of the changes underlying Model IV, the results of the final year of the simulation for the spread periods 5, 20 and 40 years are compared. Firstly, the differences between the fund distribution percentiles and the median for the three portfolios and spread periods are compared in tables 7.3.2a to 7.3.4b

5 Year Spread Period

Table 7.3.2a -The Difference Between the Median and the Percentiles of the Fund in the Final Year of the Simulation.

% of Equities in Portfolio	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
100	22.68	1.82	2.18	3.40	4.35
85	23.32	1.69	1.86	2.96	3.65
70	24.07	1.47	1.49	2.68	3.17

Table 7.3.2b - The Difference Between the Median and the Percentiles of the Fund in the Final Year of the Simulation.

% of Equities in Portfolio	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
100	22.68	4.32	5.73	5.96	7.90
85	23.32	3.68	4.94	5.26	7.25
70	24.07	3.32	4.10	4.60	6.10

20 Year Spread Period

Table 7.3.3a - The Difference Between the Median and the Percentiles of the Fund in the Final Year of the Simulation.

% of Equities in Portfolio	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
100	23.92	3.43	4.38	6.03	8.98
85	24.53	3.14	3.90	5.57	7.64
70	25.56	2.74	3.17	4.94	6.45

Table 7.3.3b - The Difference Between the Median and the Percentiles of the Fund in the Final Year of the Simulation.

% of Equities in Portfolio	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
100	23.92	7.52	12.08	10.18	20.14
85	24.53	6.79	10.44	8.70	16.98
70	25.56	6.18	8.59	8.27	13.56

40 Year Spread Period

Table 7.3.4a - The Difference Between the Median and the Percentiles of the Fund in the Final Year of the Simulation.

% of Equities in Portfolio	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
100	25.37	5.43	7.95	9.20	17.86
85	26.55	5.40	7.26	8.81	14.93
70	28.52	4.90	5.37	8.45	11.91

Table 7.3.4b - The Difference Between the Median and the Percentiles of the Fund in the Final Year of the Simulation.

% of Equities in Portfolio	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
100	25.37	11.38	24.27	14.17	44.71
85	26.55	10.69	20.45	13.03	33.46
70	28.52	10.15	16.44	12.57	27.13

From Tables 7.3.2a through to 7.3.4b, it can be seen that, for each of the three selected spread periods, increasing the proportion of indexed-linked gilts in the portfolio increases the value of the long term median, and decreases the difference between the median and all the selected percentiles. Therefore, if these differences were expressed as a percentage of the long term median this characteristic of having a less dispersed distribution of the fund would be even more emphasised. This feature occurs, as one would expect, because the standard deviation of the fund has been reduced as the proportion of indexed-linked gilts increases.

It is also worth noting that the reduction in the difference between the median and the higher percentile of each pair appears greater than that of the lower percentile. For example, for the 40 year spread period, the difference between the median and the 99% percentile decreases from 44.71 for 100% equities to 27.13 when the portfolio comprises 70% equities and 30% indexed-linked gilts. In contrast, the difference between the 1% percentile and the median only changes from 14.17 to 12.57 for the same two portfolios. These differences in the reduction are mirrored for all pairings and spread periods, with the higher spread periods having a more obvious difference between the higher and lower percentile changes. The implication of this is that the distribution of the fund appears to have changed with the introduction of indexed-linked gilts, so that it has become much less likely that extreme over-funding occurs when compared to the 100% equity portfolio.

The differences between the median and the percentiles of the fund suggest that the distribution of the fund has a log-normal shape (this is examined further in section 13).

The result of switching from a portfolio of 100% equities to one containing 70% equities and 30% indexed-linked gilts therefore appears to have made the distribution of the fund less skewed and more symmetric.

In Tables 7.3.5, 7.3.6 and 7.3.7, the changes that have occurred in the frequencies with which the fund in the final year passing through the set barriers for the spread periods 5 years, 20 years and 40 years are recorded.

Spread Period of Five Years

Table 7.3.5 - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries.

% of Eqs	% of Optimum Starting Fund							
	80%	85%	90%	95%	105%	110%	115%	120%
100%	4.70%	11.35%	21.85%	36.55%	33.70%	22.20%	13.55%	8.05%
85%	2.45%	6.80%	18.15%	35.00%	31.70%	18.40%	9.65%	5.45%
70%	0.55%	3.95%	13.05%	29.85%	28.50%	15.65%	7.50%	3.15%

Spread Period of Twenty Years

Table 7.3.6 - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries.

% of Eqs	% of Optimum Starting Fund							
	80%	85%	90%	95%	105%	110%	115%	120%
100%	12.20%	19.30%	25.90%	34.95%	49.50%	41.30%	34.05%	28.50%
85%	9.50%	15.35%	22.95%	31.75%	49.00%	40.20%	33.05%	26.20%
70%	4.55%	9.40%	16.10%	25.85%	52.00%	42.45%	32.40%	23.20%

Spread Period of Forty Years

Table 7.3.7 - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries.

% of Eqs	% of Optimum Starting Fund							
	80%	85%	90%	95%	105%	110%	115%	120%
100%	17.80%	22.70%	27.30%	35.55%	56.50%	50.85%	46.45%	41.80%
85%	13.45%	19.25%	24.95%	30.00%	58.90%	53.05%	48.25%	43.45%
70%	7.55%	11.25%	16.45%	21.40%	65.60%	59.00%	53.80%	47.70%

Table 7.3.5 records the effect of changing the percentage of indexed-linked gilts in the portfolio when the spread period is 5 years. The table shows that as the percentage of indexed-linked gilts in the portfolio increases, the frequency that the fund in the final year passes through each of the specified fund levels falls. It can also be observed that the relative reduction in the frequencies is greater for the more extreme values. For example, the frequency of the fund falling below the 80% fund level decreases from

4.70% to 0.55%, a reduction of 88%, whereas the frequency of the fund falling below the 95% fund level decreases from 36.55% to 29.85% a reduction of only 18%.

It can be seen from the frequencies for the upper fund levels that, like the lower fund levels, the reduction in frequencies is greater the more extreme is the level of the fund that we are examining. However, it should be noted that the reduction in frequencies for the upper fund levels is lower than that for the corresponding lower levels i.e. for the fund being lower than 80% the reduction in frequency between the portfolio of 100% equities and the portfolio of 70% equities and 30% ILGs is 88% whereas the reduction for the frequency of the fund being over 120% (the extreme upper fund level) is only 61%. The introduction of indexed linked gilts, therefore, appears to have lowered the variance of the distribution of the fund but the distribution has shifted so that the chances of over-funding have increased compared to the chances of under-funding.

This shift towards over-funding is clearer when the frequency levels for the 20 year spread period are examined (Table 7.3.6). Like the 5 year spread period, the 20 year spread period has seen a reduction in the frequency of the final fund being below all the given levels of the starting fund with the reductions once again being greater the more extreme the fund level. However, from the frequencies of the upper levels, we can see that the frequencies for all the levels are reduced when the portfolio changes from 100% equities to 85% equities but when the portfolio is 70% equities and 30% ILGs the frequencies are increased for the levels 105% and 110%.

Finally, Table 7.3.7 considers the case of the forty year spread period and shows that the frequencies for the lower fund levels once again decrease as the percentage of ILGs increase but the frequencies for all the upper fund levels increase. For the forty year spread period, we can see that although the introduction of ILGs decreases the standard deviation of the fund, it increases the chances of over-funding.

There appears to be some contradiction in the results obtained from Tables 7.3.2 - 7.3.4 and those from Tables 7.3.5 - 7.3.7 as the first set of tables state that over-funding is reduced more than under-funding whereas the second set of tables gives the opposite result. However, most of these contradictions can be explained by looking at the changes in the median between the portfolios. The most extreme contradiction occurs for the forty year spread period where Tables 7.3.4a and b show major reductions in the differences between the median and the higher percentiles when moving from a 100% equity portfolio to a 70% equity 30% indexed-linked gilt portfolio while Table 7.3.7 shows that the frequency of over-funding has increased. The reason for this apparent contradiction lies in the fact that, for the percentiles which are close to the median, the reduction in the gap between the percentile and the median is less than the increase in the value of the median. For example, Table 7.3.4a shows that, for the forty year spread period, the reduction in the difference between the median and the 75th percentile between the two portfolios is 2.58 while the value of the median has increased by 3.15 and so the value of the 75th percentile fund is in fact greater for the 70% equity 30% indexed-linked gilt portfolio than for the 100% equity portfolio and so the frequency of over-funding increases.

If the frequency of much higher over-funding were recorded, i.e. levels greater than 120%, then the frequencies would be less for the 70% equities 30% indexed-linked stock compared to the 100% equity portfolio, as the reduction in the difference between the median and the very high percentiles is greater than the increase in the median. This is the reason why the twenty year spread period has seen an increase in the frequencies for the 105% and 110% boundaries and a decrease in the frequencies for the 115% and 120% boundaries.

The reason why the fund has become more prone to over-funding stems from the conclusion that was reached at the end of section 6.3. It was stated there that during times of high inflation the increase in pension liabilities is capped but that these times of high inflation are usually accompanied by high equity returns which lead to over-funding. This over-funding cannot be brought under control as easily for the larger spread periods and so there is an increased frequency of over-funding in the final years of the simulations. This problem is made more acute with ILGs as they are designed to give real returns and hence, during periods of high inflation, the gap between the effective inflation rate (i.e. the rate used to increase the value of the pensions in payment) and the return on ILGs increases and so over-funding occurs.

8 Model V - Inclusion of Conventional Gilts.

8.1 Description of Model V.

In Model IV, the portfolio of the pension scheme was changed to allow investment in Indexed-Linked Gilts. For Model V, the pensions portfolio is once again split between two types of assets but this time the two classes are equities and conventional gilts. Apart from this change in the class of gilt investment, Model V is identical to Model IV.

8.2 Modelling of the Returns of Conventional Gilts.

Again, the returns on conventional gilts are modelled by the Wilkie Model (1995 version) with the standard deviation parameter halved. However, there is no model for conventional gilts so the model for Consols was used instead. The model for the yield on consols, C(t), is as follows:-

$$C(t) = CW.CM(t) + CMU.\exp\{CN(t)\}$$

where $$CM(t) = CD.I(t) + (1 - CD).CM(t - 1)$$

and $$CN(t) = CA1.CN(t - 1)$$

and $$CE(t) = CSD.CZ(t)$$

and $$CZ(t) \sim \text{i.i.d. } N(0,1)$$

The parameter values used are:-

CW = 1
CMU = 3.09%
CA1 = 0.9234
CD = 0.045
CSD = 0.096

Total nominal returns at time t, CR(t), are defined as:-

$$CR(t) = CR(t - 1).\{1/C(t) + 1\}.C(t - 1)$$

8.3 Results.

The assumed rate of interest for a portfolio of conventional gilts was obtained by finding the long term return when there were no random errors present. This gives an annual rate of return of 7.82%.

The model was run for 2000 simulations and for the same spread periods as Model IV. The model tested had a portfolio of 85% equities and 15% conventional gilts. This gives an assumed rate of return of 10.41%, with a starting fund of 23.8212 and a contribution rate of 6.61%. It should be noted that, because the assumed rate of return for conventional gilts is less than that of ILGs, the starting fund and contribution rate for a portfolio containing 85% equities and 15% conventional gilts are greater than for the portfolio containing 85% equities and 15% indexed-linked gilts. For the purpose of examining the differences between the two types of mixed portfolio, the results of the portfolio containing indexed-linked gilts are the results from Model IV for the portfolio of 85% equities and 15% indexed-linked gilts.

Table 8.3.1 - The Standard Deviations of the Extra Contribution and Fund for the Final Year of the Simulation for Models IV and V.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)		Standard Deviation of the Fund in the Final Year	
	Model V	Model IV	Model V	Model IV
5	8.00	7.77	2.78	2.65
7	6.75	6.54	3.17	3.01
10	5.86	5.71	3.72	3.55
15	5.32	5.25	4.63	4.48
20	5.27	5.26	5.61	5.47
25	5.49	5.48	6.71	6.55
30	5.85	5.83	7.91	7.70
40	6.76	6.68	10.45	10.08

Table 8.3.1 shows the standard deviations of both the fund and the extra contribution in the final year of simulation for all of the tested spread periods. There are two main things to note from this table. Firstly, it can be seen that the standard deviations for both the fund and the extra contribution are greater for Model V than for Model IV for all the spread periods which would indicate that the portfolio containing conventional gilts has a greater standard deviation than that of the indexed-linked gilt portfolio. However, the second point to note from the above table is that the critical spread period, M^*, for Model V is 20 years whereas it is 15 years for Model IV. According to the theoretical results (see section 2) we would expect Model V to have the shorter critical spread period, if it has a larger standard deviation - so there appears to be a contradiction.

There are a number of potential reasons why this unexpected result may have occurred. Firstly, the last year of the simulation is the one that has been used to record the long-term standard deviations. For Model IV, if another year had been picked close to the 149th year then the results would indicate that the critical spread period for Model IV is also 20 years as for Model V. It must also be considered that equities still represent a far greater proportion of the portfolio than conventional gilts and so the volatility of the returns of the equities will also make some difference to the results because of sampling errors.

Tables 8.3.2 a-c show the frequency of over- and under-funding for Models IV and V for the spread periods 5, 20 and 40 years.

Spread Period of Five Years

Table 8.3.2a - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries for Models IV and V.

	% of Optimum Starting Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
IV	2.45%	6.80%	18.15%	35.00%	31.70%	18.40%	9.65%	5.45%
V	2.65%	8.30%	18.00%	33.50%	34.20%	21.15%	11.20%	4.90%

Spread Period of Twenty Years

Table 8.3.2b - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries for Models IV and V.

	% of Optimum Starting Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
IV	9.50%	15.35%	22.95%	31.75%	49.00%	40.20%	33.05%	26.20%
V	9.70%	15.55%	21.80%	30.90%	51.40%	43.05%	34.65%	27.95%

Spread Period of Forty Years

Table 8.3.2c - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries for Models IV and V.

	% of Optimum Starting Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
IV	13.45%	19.25%	24.95%	30.00%	58.90%	53.05%	48.25%	43.45%
V	13.20%	17.75%	21.80%	28.05%	60.95%	55.55%	50.35%	45.30%

Table 8.3.2c

The changes in the frequencies with which the fund passes through the designated boundaries for the five year spread period are recorded in Table 8.3.2a. For the higher boundaries, there has been a definite increase in the frequencies when the model changes from Model IV to Model V for all but the 120% barrier where the frequency decreases by 0.55%. For the lower boundaries, there is a decrease for the 95% boundary of 1.50%, while there is an increase for the 85% boundary of the same magnitude. The 90% and 80% boundaries record a negligible fall and rise respectively in their frequencies. Overall, it therefore appears that the fund for Model V is more volatile than that of Model IV even when scaling to allow for the size of fund is taken into account.

From Table 8.3.2b, it can be seen that all of the higher fund boundaries experience an increase in their frequencies when the model is changed from Model IV to Model V and there has been slight, statistically negligible increase of 0.2% in the frequencies for the 80% and 85% boundaries. The 90% and 95% boundaries, however, have seen a fall in their frequencies of 1.15% and 0.85%. If these decreases in frequencies are compared with the increases in frequencies for the 105% and 110% boundaries which are 2.40% and 2.85% respectively, then there is strong evidence that the 20 year spread period has a more volatile fund for Model V compared to Model IV.

Table 8.3.2c shows that, for the forty year spread period, the higher boundaries have all seen an increase in their frequencies ranging in size from 1.85% for the 120% boundary to 2.50% for the 110% boundary. The lower boundaries have all seen decreases in their frequencies ranging from 3.15% for the 90% boundary to 0.25% for the 80% boundary. Again, the fund using a forty year spread period appears to have become more volatile although the evidence from examining the frequencies is not conclusive.

Conventional gilts, therefore, appear to give a more unstable fund than indexed-linked gilts although the evidence is not totally conclusive. However, because conventional gilts give a lower return than indexed-linked gilts and there is certainly no evidence that the fund becomes more stable, it has been decided not to pursue an investigation into a three asset model containing equities, indexed-linked gilts and conventional gilts as indexed-linked gilts appear to be a 'better' asset (i.e. lead to more efficient portfolios) than conventional gilts when using the Wilkie Model.

In this regard, we note the conclusions of Smith (1996) and Huber (1997), when reviewing the properties of the Wilkie Model, that ILGs have a higher expected real return and a lower standard deviation when compared to the long-term fixed interest asset class (and an approximately similar covariance structure). "Hence, there appears to be little incentive to invest in the long-term fixed interest asset" class. Huber (1997) also concludes that this component of the Wilkie Model suffers from empirical inadequacy.

9 Model VI - Different Rates of Return Assumptions.

9.1 Description of Model VI.

At the end of section 8.3, it was concluded that, given the opportunity to split funds between three classes of asset (equities, ILGs, Consols), the most satisfactory portfolio to use for the model was a mixed portfolio of equities and indexed-linked gilts, where increasing the proportion of ILGs leads to increased stability but also an increased standard contribution. However, for the longer spread periods there was still a significant trend towards over-funding due to the fact that the model uses LPI for uprating pensions in payment but there had been no re-adjustment in the assumptions for the rates of return for equities and indexed-linked gilts.

It was decided, therefore, that the current model (reverting back to Model IV) should be stabilised before it was adapted further especially as the adaptations for Model VII (see section 10) and Model VIII (see section 11) would create a more volatile model. As a stable model was desired, the model developed here has a portfolio of 70% equities and 30% indexed-linked gilts.

In order to make the model as stable as possible, the new assumed rates of return were selected to give a long-term extra contribution rate that had a slightly negative mean and a slightly positive median. The extra contribution rate has been targeted instead of the fund, because the presence of LPI for the pensions in payment tends to lead to the desired fund (in real terms) in later years being less than the starting fund. This is because the value of the pensions in payment is dependent on the gap between wage inflation and the value of the increase in pensions. During years of high inflation, the real value of the pensions in payment drops (assuming that the difference between wage and price inflation remains relatively constant) as the indexation is capped at 5% and so the gap between the annual wage increase and the amount by which the pension is uprated increases. Therefore, the later years of the simulation tend to show a smaller desired funding level, because there is a larger probability that a period of high inflation has occurred.

The assumed rates of return that were chosen are 11.22% for equities and 9.15% for indexed-linked gilts so the assumed rates of return for the portfolio was 10.60%. This gives a starting fund of 23.4304 and a contribution rate of 6.30%.

The model was run for 2000 simulations with the spread periods being 5, 7, 10, 15, 20, 25, 30 and 40 years. It should be noted that the returns generated by these simulations will be used by each of the subsequent models in order to facilitate the comparisons of the results since then some of the sampling errors will be removed. Thus, although we have a sample of 2000, it is likely that each set of 2000 simulations will produce rates of return of differing standard deviations. When comparing two models that are using different sets of generated returns, changes in the standard deviation of the extra contribution arising in the two models may be attributed to the differences in the models but may actually be caused by the differences in the standard deviations of the two sets of generated returns. Using the same set of generated returns to test both models should, therefore, remove this type of sampling error.

9.2 Results.

In the following discussion of the effects of the new assumed rates of return, the results using the new rates of return are presented as Model VI and the Model IV results are those obtained using the assumed rates of return when the portfolio was invested as 70% equities and 30% indexed-linked gilts (as discussed in section 7).

Table 9.2.1 shows the standard deviation of the fund and the extra contribution in the final year of simulation for Models IV and VI. The first thing to note from the table is that the standard deviation of both the extra contribution and the fund is less for each spread period when using the new assumed rates of return compared to the results from the original model. It should also be noted that this difference between the standard deviations increases as the spread period increases.

Table 9.2.1 - The Standard Deviations of the Extra Contribution and Fund for the Final Year of the Simulation for Models IV and VI.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)		Standard Deviation of the Fund in the Final Year	
	Model VI	Model IV	Model VI	Model IV
5	6.41	6.66	2.17	2.30
7	5.35	5.60	2.45	2.61
10	4.57	4.84	2.82	3.05
15	4.03	4.37	3.41	3.79
20	3.85	4.31	3.99	4.59
25	3.84	4.45	4.56	5.46
30	3.92	4.70	5.15	6.41
40	4.19	5.37	6.28	8.42

The second point of interest is that the critical spread period, M^*, has increased from 20 years to 25 years which, according to the theoretical results of section 2, would appear to indicate that the model is indeed more stable with the newer assumed rates of return. Of course, the fall in standard deviation from year 20 to year 25 for Model VI is only 0.01 but this compares to an increase in standard deviation of 0.14 for Model IV. So it seems reasonable to conclude that the critical spread period has been reduced, taking into account any errors from not having a larger number of simulations, since both models are using the same generated annual rates of return.

It was stated in section 9.1 that the motivation for selecting the new assumed rates of return was to stabilise the long-term mean and median for the fund and extra contribution.

Table 9.2.2 - The Mean and Median of the Fund for Models IV and VI in the Final Year of the Simulation.

Spread Period (years)	Mean of Fund in the Final Year		Median of the Fund in the Final Year	
	Model VI	Model IV	Model VI	Model IV
5	23.07	24.20	22.96	24.07
20	23.13	26.04	22.73	25.60
40	23.30	29.57	22.27	28.52

Table 9.2.2

Table 9.2.2 shows how the new assumed rates of return have meant that the increase in the mean of the fund when moving from the 5 year spread period to the 20 and 40 year spread periods has been drastically reduced and the median now decreases slightly as the spread period increases.

Tables 9.2.3a to 9.2.5b show the distribution of the fund for three selected spread periods by showing the difference between the median and the selected percentiles of the distribution of the fund in the final year of the simulation for both Model IV and Model VI

__5 Year Spread Period__

Table 9.2.3a -The Difference Between the Median and the Percentiles of the Fund for Models IV and VI.

Model	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
VI	22.96	1.47	1.56	2.58	2.95
IV	24.07	1.47	1.49	2.68	3.17

Table 9.2.3b -The Difference Between the Median and the Percentiles of the Fund for Models IV and VI.

Model	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
VI	22.96	3.11	3.80	4.23	5.76
IV	24.07	3.32	4.10	4.60	6.10

20 Year Spread Period

Table 9.2.4a -The Difference Between the Median and the Percentiles of the Fund for Models IV and VI.

Model	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
VI	22.73	2.47	2.69	4.32	5.60
IV	25.56	2.74	3.17	4.94	6.45

Table 9.2.4b -The Difference Between the Median and the Percentiles of the Fund for Models IV and VI.

Model	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
VI	22.73	5.23	7.64	7.01	11.64
IV	25.56	6.18	8.59	8.27	13.56

40 Year Spread Period

Table 9.2.5a -The Difference Between the Median and the Percentiles of the Fund for Models IV and VI.

Model	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
VI	22.27	3.42	4.36	5.92	9.21
IV	28.52	4.90	5.37	8.45	11.91

Table 9.2.5b -The Difference Between the Median and the Percentiles of the Fund for Models IV and VI.

Model	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1%	99%
VI	22.27	7.36	12.51	9.44	20.86
IV	28.52	10.15	16.44	12.57	27.13

It can be observed that the differences between the median and each percentile for a particular spread period is smaller for Model IV than for Model VI (with the exception of the 25% and 75% percentiles for the five year spread period which can probably be accounted for by sampling errors). This result is as expected because Table 9.2.1 shows that the standard deviation for each spread period has been reduced. Also, the reduction in the differences of the percentiles has been reduced more for the larger spread periods, which again corroborates the results obtained from Table 9.2.1.

Table 9.2.6 - Relative Changes Between Model VI and Model IV of the Difference Between the Median and Each Percentile.

Spread Period (years)	The Difference Between Percentile and Median From Model VI Expressed as a Percentage of the Difference From Model IV							
	25%	75%	10%	90%	5%	95%	1%	99%
5	1.000	1.047	0.962	0.931	0.937	0.927	0.920	0.944
20	0.901	0.849	0.874	0.868	0.846	0.889	0.847	0.858
40	0.698	0.812	0.701	0.773	0.725	0.761	0.751	0.769

Table 9.2.6 shows the difference between the median and each percentile for Model VI expressed as a percentage of the corresponding value of Model IV. This Table clearly indicates that (as already stated) the reductions in the differences are greater for the longer spread periods. Because of the effects of sampling errors due to the small number of simulations, there are no conclusive trends in the reductions of any given spread period. However, it should be noted that, for the spread period of forty years, the reduction in each pair of percentiles is less for the higher percentile and that for the 1% and 99% percentiles each spread period has seen a larger reduction for the 1% percentile.

Finally, Tables 9.2.7, 9.2.8 and 9.2.9 below show the frequencies with which the fund in the final year passes through the specified boundaries for the spread periods of five, twenty and forty years.

<u>Spread Period of Five Years</u>

Table 9.2.7 - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries for Models IV and VI.

Model	% of Optimum Starting Fund							
	80%	85%	90%	95%	105%	110%	115%	120%
VI	1.05%	5.80%	18.75%	38.30%	24.05%	11.00%	4.10%	1.45%
IV	0.55%	3.95%	13.05%	29.85%	28.50%	15.65%	7.50%	3.15%

<u>Spread Period of Twenty Years</u>

Table 9.2.8 - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries for Models IV and VI.

Model	% of Optimum Starting Fund							
	80%	85%	90%	95%	105%	110%	115%	120%
VI	12.60%	21.45%	33.35%	44.75%	31.50%	22.40%	15.90%	10.55%
IV	4.55%	9.40%	16.10%	25.85%	52.00%	42.45%	32.40%	23.20%

Table 9.2.8

Spread Period of Forty Years

Table 9.2.9 - Frequency that the Fund in the Final Year Passes Through the Specified Boundaries for Models IV and VI.

	% of Optimum Starting Fund							
Model	80%	85%	90%	95%	105%	110%	115%	120%
VI	24.25%	32.65%	40.65%	49.95%	35.60%	28.90%	23.55%	18.90%
IV	7.55%	11.25%	16.45%	21.40%	65.60%	59.00%	53.80%	47.70%

Table 9.2.9

It appears from the tables that the distribution of the fund in the final year has been transformed from one of over-funding to one of under-funding with the change of model from Model IV to Model VI. Indeed, each spread period has seen a rise in the frequency for all the lower boundaries and a decrease in the frequencies for all of the upper boundaries to such an extent that the frequency for each lower boundary is now greater than for the corresponding higher boundary i.e. the frequency for the 95% fund boundary for the forty year spread period is 49.95% compared to 35.60% for the 105% fund boundary.

However, as was mentioned in section 9.1 in the discussion of the selection of the new parameters, the desired fund level in real terms is not the same for each year and, because of LPI, the long term desired fund level in real terms is lower than the starting fund level. Hence, the frequencies with which the lower boundaries are breached are smaller and the corresponding frequencies for the higher boundaries are greater than those stated if true over and under-funding were considered.

10 Model VII - Triennial Valuations.

10.1 Description of Model VII.

So far, each model which we have tested has included annual valuations to determine the amount of extra contribution required. In practice, this would not always be the case because of the expenses involved and the minimum requirements set down in UK regulations. To reflect this feature, we have changed the frequency of valuations and Model VII has valuations every three years rather than every year.

Model VII is identical to Model VI in every aspect apart from the valuations occurring every three years. The extra contribution that is determined at each valuation is now paid for the next three years until the next valuation. The monetary amount of the contribution is linked to wage inflation so that the extra contribution expressed as a percentage of salary is constant for the three years. This can be seen in the detailed results generated for this model (not shown here), where the extra contribution is the same for years 0 to 2, 3 to 5 and 6 to 8 for any spread period.

Model VII not only has the same asset allocation as Model VI but was tested using the same rates of return as Model VI so that comparisons between the two models would be straightforward (see section 9). For example, as each model has been simulated 2000 times generating its own rates of return for each year, it is possible that the last ten years of returns would be more varied for one model than the other and so some conclusions drawn on the differences between the models would be affected by this. As Model VII has the same variance in the generated returns as Model VI, these sampling errors have been removed. Model VII has thus been simulated 2000 times for 149 years and for the spread periods 5, 7, 10, 15, 20, 25, 30 and 40 years.

10.2 Theoretical Results.

The theoretical results for the triennial valuations with the IID investment return model are derived in Section 3 of the appendix. The two main results are that compared to annual valuations, triennial valuations lead to a higher standard deviation of both the fund and the extra contribution when all other parameters, such as the length of spread period and the standard deviation of the returns, remain unchanged.

The second result is that the optimal spread period for triennial valuations is slightly larger than for annual valuations. Table 10.2.1 shows the theoretical optimal spread period for different values of the mean and variance of the investment returns, based on the IID model for investment returns (Haberman (1993)).

Comparison of Optimal Values of the Spread Period M When Valuations are Annual (M^*) and Triennial (M^*_1).

	i					
	1%		3%		5%	
σ	M^*	M^*_1	M^*	M^*_1	M^*	M^*_1
0.05	60	61	23	24	14	15
0.1	42	43	20	21	13	14
0.15	28	29	16	17	11	13
0.2	19	20	13	14	10	11
0.25	14	15	10	11	8	9

Table 10.2.1

Table 10.2.1 clear shows that, although the spread period is increased when the valuations are changed from annual to triennial, the difference is very small being approximately one year only in most cases.

10.3 Results.

Table 10.3.1 shows the standard deviation of the extra contribution and fund in the final year of simulation for Model VII (triennial valuations) and Model VI (annual valuations).

Table 10.3.1 - The Standard Deviations of the Extra Contribution and Fund for the Final Year of Simulation for Models VI and VII.

Spread Period (years)	Standard Deviation of the Extra Contribution in the Final Year (%)		Standard Deviation of the Fund in the Final Year	
	Model VII	Model VI	Model VII	Model VI
5	7.70	6.41	2.44	2.17
7	6.01	5.35	2.68	2.45
10	4.90	4.57	3.02	2.82
15	4.23	4.03	3.58	3.41
20	4.03	3.85	4.15	3.99
25	4.01	3.84	4.73	4.56
30	4.10	3.92	5.32	5.15
40	4.39	4.19	6.47	6.28

Table 10.3.1 shows that the standard deviation of both the fund and the extra contribution for any spread period is greater for Model VII than for Model VI, which is in agreement with the first theoretical result. This result is reasonable and can be explained as follows. The contribution paid in the last two years of every three year period is not based on the current funding level but on the funding level that occurred one or two years ago. Hence, extra contributions may be being paid when there is already over-funding or there may be a negative extra contribution at a time of under-

funding. Naturally, this could lead to a larger positive or negative extra contribution at the next valuation compared to the case of annual valuations.

It should also be noted that the relative increase in the standard deviations is greater for the smaller spread periods. For example, the standard deviation of the fund for the five year spread period has increased by 12% between the two models whereas the increase for the forty year spread period is only 3%. Again, this result is as expected since paying three years of a contribution based on a five year spread period will have a greater effect than paying three years of a contribution based on a forty year spread period.

Finally, although Model VII has a greater variance than Model VI the critical spread period, M^*, has remained at 25 years which agrees to some extent with the second theoretical result as Table 10.2.1 showed that the spread period only increases by one year. As the simulation has been performed using only the spread periods 20, 25 and 30 years rather than every integral spread period length between 20 and 30 years, a small change in the length of the optimal spread period would not have been detected.

Table 10.3.2 shows the mean and median values of the fund in the final year of the simulation for the spread periods 5, 20 and 40 years for Model VII and Model VI.

Table 10.3.2 - The Mean and Median of the Fund for Models VI and VII in the Final Year of the Simulation.

Spread Period (years)	Mean of Fund in the Final Year		Median of the Fund in the Final Year	
	Model VII	Model VI	Model VII	Model VI
5	23.07	23.07	22.89	22.96
20	23.08	23.13	22.61	22.73
40	23.21	23.30	22.12	22.27

It can be seen that both the mean and the median values of the fund are less for Model VII than Model VI (except for the mean of the 5 year spread period which remains unchanged), with the difference becoming greater as the length of the spread period increases. Also, the size of the median falls to a greater extent than that of the mean for each spread period.

Tables 10.3.3a - 10.3.5b show the differences between the median and the values of the selected percentiles for Models VI and VII with the spread periods 5, 20 and 40 years.

5 Year Spread Period

Table 10.3.3a -The Difference Between the Median and the Percentiles of the Fund for Models VI and VII.

Model	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
VII	22.89	1.57	1.80	2.87	3.43
VI	22.96	1.47	1.56	2.58	2.95

Table 10.3.3b -The Difference Between the Median and the Percentiles of the Fund for Models VI and VII.

Model	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1st	99%
VII	22.89	3.52	4.36	4.63	6.63
VI	22.96	3.11	3.80	4.23	5.76

20 Year Spread Period

Table 10.3.4a -The Difference Between the Median and the Percentiles of the Fund for Models VI and VII.

Model	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
VII	22.60	2.55	2.86	4.46	5.92
VI	22.73	2.47	2.69	4.32	5.60

Table 10.3.4b -The Difference Between the Median and the Percentiles of the Fund for Models VI and VII.

Model	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1st	99%
VII	22.60	5.37	7.86	7.03	12.19
VI	22.73	5.23	7.64	7.01	11.64

40 Year Spread Period

Table 10.3.5a -The Difference Between the Median and the Percentiles of the Fund for Models VI and VII.

Model	Median	Difference Between the Median and the Respective Percentile			
		25%	75%	10%	90%
VII	22.12	3.55	4.37	5.94	9.65
VI	22.27	3.42	4.36	5.92	9.21

Table 10.3.5b -The Difference Between the Median and the Percentiles of the Fund for Models VI and VII.

Model	Median	Difference Between the Median and the Respective Percentile			
		5%	95%	1st	99%
VII	22.12	7.45	12.94	9.53	21.67
VI	22.27	7.36	12.51	9.44	20.86

Table 10.3.5b

Tables 10.3.3a to 10.3.5b show that the difference between the median and each of the selected percentiles is greater for Model VII than Model VI for all the spread periods. This result is as expected, because Table 10.3.1 had shown that the standard deviation of the fund has increased for all spread periods when triennial valuations are introduced.

Tables 10.3.3a to 10.3.5b also show that the relative increases in the differences between the median and the selected percentiles are noticeably greater for the five year spread period than for the twenty and forty year spread periods. This trend once again supports the results obtained in Table 10.3.1, as the increase in the standard deviation of the fund for the five year spread period was 12% compared to 4% for the twenty year spread period and 3% for the forty year spread period

From a comparison of pairs of the percentiles e.g. the 25% and 75% percentiles, it can be seen that the relative increase in the difference between the median and the percentiles is always greater for the higher percentile of each pair (with the exception of the 25% and 75% percentiles for the forty year spread period).

Tables 10.3.6, 10.3.7 and 10.3.8 show the frequencies with which that the extra contribution passes through the defined boundaries in the final year of the simulation for the spread periods 5, 20 and 40 years respectively for both Model VI and Model VII. An examination of the frequencies for the extra contribution is of greater benefit than looking at the fund frequencies as the value of the extra contribution is determined by the desired funding level in the year of the valuation. Therefore, a negative extra contribution means that the fund is greater than the actuarial liability although the size of fund itself may be smaller than the initial actuarial liability, and so over and under-funding are more accurately recorded. Unlike comparisons between previous models where the frequencies of the fund size have been investigated, both Models VI and VII have the same standard contribution rate and so comparisons are not distorted by the size of the standard contribution rate. However, it must be remembered that the contribution for the final year of the simulation, year 149, for Model VII was calculated in year 147 so the rates of return for the years 148 and 149 are not taken into account when determining extra contribution rates for Model VII.

Spread Period of Five Years

Table 10.3.6 - Frequency that the Extra Contribution of the Final Year Passes Through the Specified Boundaries For Models VI and VII.

	Extra Contribution as % of Standard Contribution							
Model	-150%	-100%	-50%	-25%	25%	50%	100%	150%
VII	10.60%	20.75%	34.75%	42.75%	41.40%	34.65%	20.55%	8.45%
VI	8.55%	17.15%	32.30%	40.35%	41.60%	32.20%	16.25%	4.65%

Spread Period of Twenty Years

Table 10.3.7 - Frequency that the Extra Contribution of the Final Year Passes Through the Specified Boundaries For Models VI and VII.

	Extra Contribution as % of Standard Contribution							
Model	-150%	-100%	-50%	-25%	25%	50%	100%	150%
VII	2.25%	6.95%	19.95%	31.55%	36.50%	21.20%	2.70%	0.00%
VI	2.30%	6.70%	20.00%	31.45%	36.15%	19.80%	2.10%	0.05%

Spread Period of Forty Years

Table 10.3.8 - Frequency that the Extra Contribution of the Final Year Passes Through the Specified Boundaries For Models VI and VII.

	Extra Contribution as % of Standard Contribution							
Model	-150%	-100%	-50%	-25%	25%	50%	100%	150%
VII	4.05%	8.65%	21.10%	30.55%	39.45%	22.20%	2.00%	0.00%
VI	3.25%	8.55%	20.40%	31.85%	37.80%	20.65%	1.95%	0.00%

Table 10.3.8

From Table 10.3.6, it can be seen that the effect of introducing triennial valuations whilst using a five year spread period is to increase the frequency for all the boundaries with the exception of the 25% of the standard contribution boundary where the frequency has decreased slightly from 41.60% to 41.20% - this may be of no real statistical significance. It should also be noted that the larger increases in the frequencies occur for the more extreme boundaries with the positive boundaries having a larger increase in frequencies than the negative boundaries. For the 25% and -25%

boundaries, the changes in frequencies between Model VI and Model VII have meant that the frequency for the -25% barrier is now greater than for the 25% barrier.

The effect of introducing triennial valuations for the twenty year spread period is different to that for the five year spread period because, although the frequencies for all the positive boundaries increase (except for the 150% barrier which falls from 0.05% to 0%), the frequencies for the lower boundaries appear to be more or less unchanged with the two boundaries showing a slight increase in their frequencies and the other two showing a slight decrease. The reason for this may be the fact that the fund for Model VII has a smaller median than that for Model VI but a slightly larger variance; hence, the changes in the median and the variance together increase the chance of under-funding which leads to an increase in the frequencies of positive extra contributions. For over-funding, the changes in the median and the variance cancel each other out and hence the frequency of the negative contribution barriers remains similar.

Table 10.3.8 shows that the effect of switching from annual to triennial valuations for the forty year spread period is once again different from the preceding two spread periods. For the positive boundaries, there has been a much greater increase in the frequency for the 25% boundary than was the case for the twenty year spread period, a similar increase for the 50% barrier compared to the twenty year spread period and a smaller increase for the 100% barrier. For the negative barriers, the frequencies for the boundaries of -150%, -100% and -50% all have increased for Model VII compared to Model VI but there has been a definite fall in frequency for the -25% barrier which neither of the other two spread periods have shown.

In conclusion, the general effect of switching from annual valuations to triennial valuations is to increase the standard deviation of both the fund and extra contribution for all spread periods. However, the magnitude and character of the changes to the distributions of the fund and extra contribution depend on the size of the spread period.

11 Model VIII - Different Initial Funding Levels.

11.1 Description of Model VIII.

The models that have been investigated so far have started from a position where the initial fund has been assumed to be equal to the actuarial liability. This, of course, will not always be the case in practice and so Model VIII is designed to test how different starting fund levels affect the variance of the fund and extra contribution both in the short and long term.

Apart from the initial funding level, Model VIII is identical to Model VI with the investment portfolio being made up of 70% equities and 30 % indexed-linked gilts. As for Model VII, the returns used for Model VIII are those that were generated for Model VI and so comparisons are easier to make regarding the effects caused by the different initial funding levels.

Model VIII was simulated 2000 times for 149 years, with the spread periods 5, 7, 10, 15, 20, 25, 30 and 40 years and from four different funding levels:- 80%, 90%, 110% and 120% of the actuarial liability.

11.2 Theoretical Results.

Returning to the IID model described in the Appendix, it is possible to investigate how the standard deviation of the extra contribution would be affected by different levels of initial funding. The two main points of interest are how the standard deviation is affected in the short-run and the differences, if any, when the model is run for a number of years that tends to infinity. Below are the theoretical results.

Then, using the notation of the Appendix,

$$Var\ C(t) = bk^2 a^t \sum_{j=1}^{t} a^{-j} \left[q^j F_0 + AL\left(1-q^j\right)\right]^2$$

where $k = \dfrac{1}{\ddot{a}_{\overline{m}|}}$

$$b = \frac{\sigma^2}{(1+i)^2}$$

$$a = (1+i)^2(1-k)^2(1+b).$$

This can be re-written as:-

$$Var\ C(t) = bk^2 a^t \sum_{j=1}^{t} a^{-j}\left[AL + (F_0 - AL)q^j\right]^2$$

Let $F_0 - AL = w$, representing the level of initial funding. Then,

$$Var\ C(t) = bk^2 a^t \sum_{j=1}^{t} a^{-j}\left[AL + wq^j\right]^2 \qquad \text{and so}$$

$$\frac{d}{dw} Var\ C(t) = 2bk^2 a^t \sum_{j=1}^{t} a^{-j} q^j \left[AL + wq^j\right]$$

So $\frac{d}{dw} VarC(t) > 0$ if $AL+wq^j > 0$ (which it is for most values of w).

Therefore, the first conclusion is that Var C(t) increases as w increases. In comparison with the fund starting at a value equal to the actuarial liability, the theoretical conclusion is that in the short-term the standard deviation of the contribution should be greater if there is an initial funding surplus and the difference should increase as the surplus increases. For a funding deficit, the standard deviation is lower and continues to decrease as the initial funding deficit is increased, so long as the condition $AL+wq^j >$ 0 holds.

The second theoretical result that needs to be derived concerns the form of

$\lim_{t \to \infty} Var\ C(t)$.

$$Var\ C(t) = bk^2 a^t \sum_{j=1}^{t}\left[AL.a^{-j} + w^2\left(\frac{q^2}{a}\right)^j + 2AL.w\left(\frac{q}{a}\right)^j\right]$$

$$= bk^2 a^t \left[\frac{\frac{1}{a}\left(1 - \frac{1}{a^t}\right)}{1 - \frac{1}{a}} AL + w^2 \frac{\frac{q^2}{a}\left(1 - \left(\frac{q^2}{a}\right)^t\right)}{1 - \frac{q^2}{a}} + 2AL.w \frac{\frac{q}{a}\left(1 - \left(\frac{q}{a}\right)^t\right)}{1 - \frac{q}{a}}\right]$$

$$= bk^2 \left[\frac{a^t - 1}{a - 1} AL + w^2 q^2 \left(\frac{a^t - q^{2t}}{a - q^2}\right) + 2AL.wq\left(\frac{a^t - q^t}{a - q}\right)\right]$$

So, Var C(t) $\rightarrow \dfrac{bk^2 AL}{1-a}$ as t $\rightarrow \infty$ independently of w, assuming that a < 1 and q < 1.

Thus, the variance of the contribution for a given spread period should tend to the same limit independently of w, i.e. regardless of the value of the initial funding level.

11.3 Simulated Results.

Our main objective in analysing the simulated results is to compare these with the results obtained from the theoretical model described in the previous section. As we are using the same attained annual rates of return from the investment model for each of the initial funding levels including the 100% initial funding level (Model VI), the differences in the results for each year should be the result of the initial funding level only rather than attributable to sampling errors.

We consider the first theoretical result, that the standard deviation of the extra contribution in the short-term increases as the funding surplus increases. Table 11.3.1 shows the standard deviation of the extra contribution in year 9 of the simulation for each of the initial funding levels (the results for the 100% funding level corresponding to those obtained for Model VI in section 9) and spread periods. Year 9 has been chosen simply as a typical year from early on in the simulations.

Table 11.3.1 - The Standard Deviation of the Extra Contribution in Year 9 of the Simulation for all Initial Funding Levels.

Spread Period	Initial Funding Level				
(years)	80%	90%	100%	110%	120%
5	5.80%	5.91%	6.03%	6.16%	6.28%
7	4.53%	4.71%	4.89%	5.07%	5.25%
10	3.51%	3.72%	3.94%	4.16%	4.37%
15	2.67%	2.89%	3.12%	3.34%	3.57%
20	2.24%	2.45%	2.67%	2.89%	3.10%
25	1.98%	2.18%	2.39%	2.60%	2.81%
30	1.81%	2.00%	2.20%	2.40%	2.60%
40	1.60%	1.79%	1.97%	2.16%	2.35%

From the results in Table 11.3.1, it is clear that the theoretical results have been demonstrated as, for each spread period, the standard deviation of the extra contribution for year 9 is greater the higher is the initial funding level. Similar conclusions follow when other choices of year are made.

We next investigate the second theoretical result, that the standard deviation of the extra contribution tends to a value which is independent of the size of the initial fund.

Table 11.3.2 shows the standard deviation of the extra contribution in year 149 of the simulation.

Table 11.3.2 - The Standard Deviation of the Extra Contribution in Year 149 of the Simulation for all Initial Funding Levels.

Spread Period	Initial Funding Level				
(years)	80%	90%	100%	110%	120%
5	6.41%	6.41%	6.41%	6.41%	6.41%
7	5.35%	5.35%	5.35%	5.35%	5.35%
10	4.57%	4.57%	4.57%	4.57%	4.57%
15	4.03%	4.03%	4.03%	4.03%	4.03%
20	3.85%	3.85%	3.85%	3.86%	3.86%
25	3.82%	3.83%	3.84%	3.85%	3.87%
30	3.85%	3.88%	3.92%	3.95%	3.98%
40	3.99%	4.09%	4.19%	4.28%	4.38%

The results in Table 11.3.2 appear to validate the second theoretical result as it can be seen that for the smaller spread periods the standard deviation of the extra contribution is the same in year 149 for each of the initial funding levels. For the larger spread periods, none of the other funding levels has the same standard deviation value as the 100% funding level.

Even for the spread periods and funding levels that have standard deviations different from the 100% initial funding level, it can be seen that the difference between the standard deviations is less than the difference between the respective standard deviations for year 9. This implies that the standard deviations are converging and if there were more years of the simulation the standard deviations would become equal.

The conclusion that the speed of convergence is determined by both the length of the spread period and the initial funding level is an expected result, as the larger is the departure from a 100% initial funding level so the time taken for the surplus/deficit to be removed will increase. Similarly, we have seen that the smaller is the spread period the more quickly the funding surpluses and deficits are removed. We would, therefore, expect that the smaller is the spread period, the larger will be the extra contributions and the more quickly the initial funding surplus or deficit will be removed and come into line with the value of the actuarial liability.

The effect that the initial funding level and spread period have on the length of time taken for the standard deviation of the extra contribution to attain the same value as for the 100% initial funding level is examined in Table 11.3.3. Table 11.3.2 leads to the conclusion that for the twenty year spread period, full convergence of the standard deviation of the extra contribution has occurred for the cases of the 110% and 120% funds but not yet for the cases of the 80% and 90% funds. However, this result is not exactly true because it is affected by rounding errors. In fact the standard deviations for the 100% and 90% initial funding cases differ by approximately 0.003% whereas the difference between the standard deviations for the 120% and 100% initial funding cases differ by approximately 0.006%. In Table 11.3.3, the definition of the year when the standard deviations fully converge is when the difference between the two standard deviations is less than 0.005% and so for the 20 year spread period there is convergence for the 90% and 110% initial fund cases but not the 80% and 120% fund cases.

Table 11.3.3 - The Year of the Simulation Where the Standard Deviation of the Extra Contribution Fully Converges.

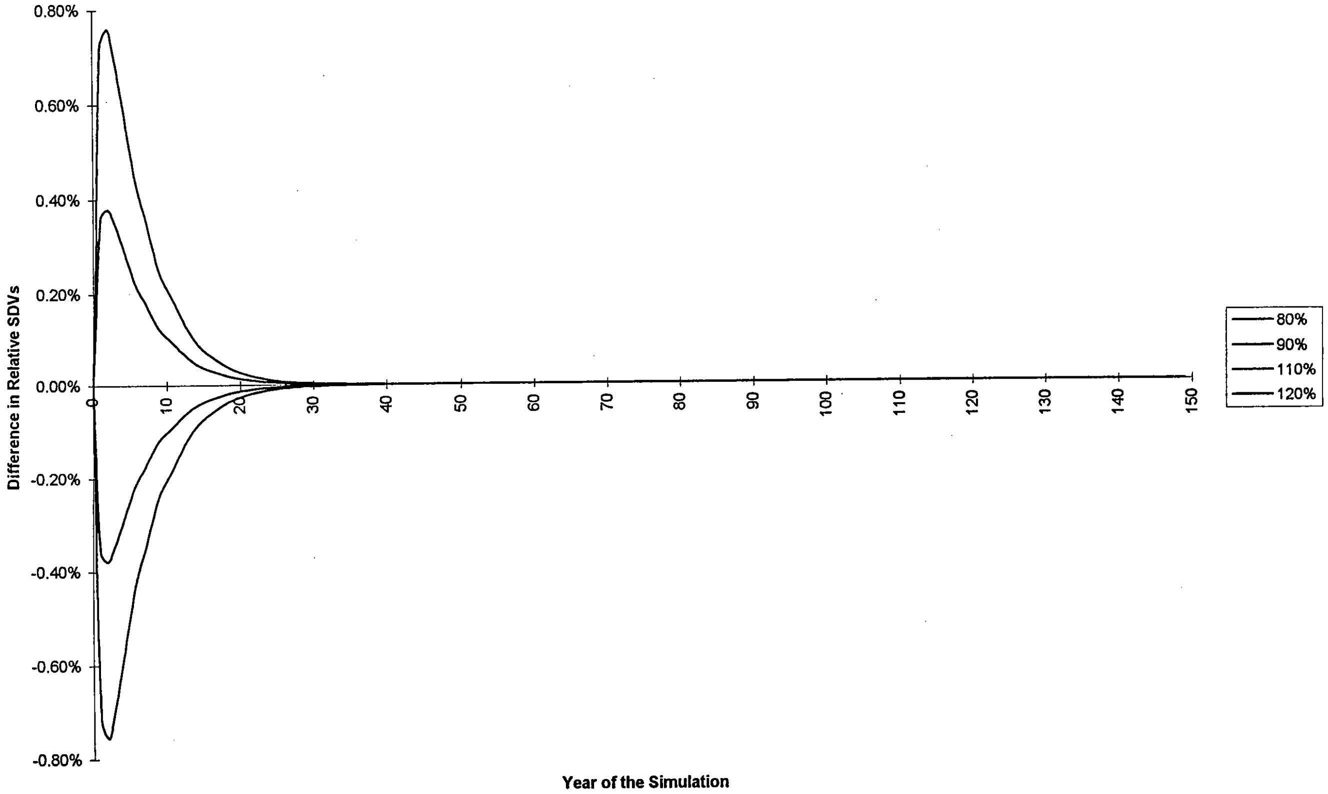
Spread Period	Initial Funding Level			
(years)	80%	90%	110%	120%
5	29	25	25	29
7	42	38	38	42
10	64	56	56	64
15	106	92	92	106
20	-	135	135	-
25	-	-	-	-
30	-	-	-	-
40	-	-	-	-

Table 11.3.3 shows the year in which the standard deviation in respect of each initial funding level equals that of the 100% funding level for the different spread periods. It is possible to ascertain how great the effect of both the spread period and the initial funding level have on the time taken for convergence by examining the rows of the Table. Similarly, examining each column shows the effect of the different spread periods for a given initial funding level.

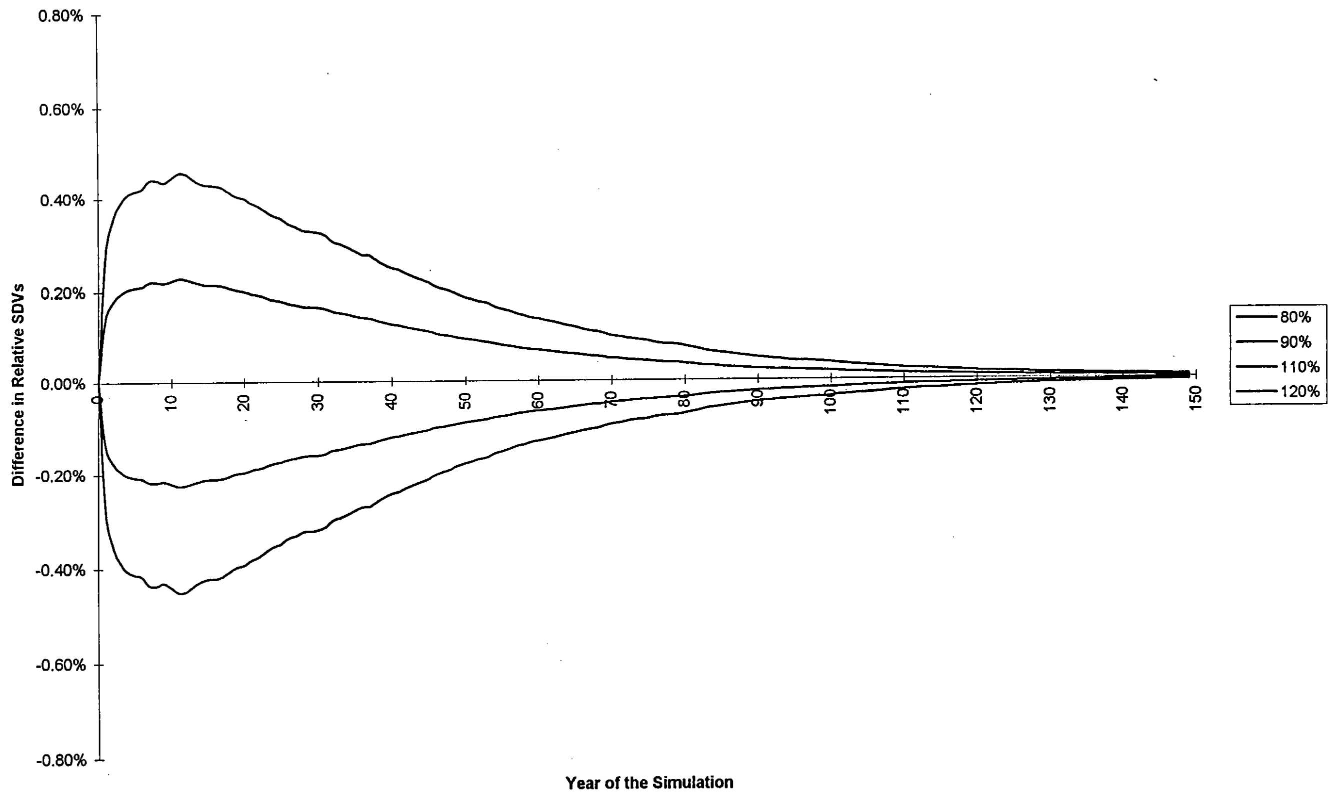
It is apparent from Table 11.3.3 that the time taken to convergence is symmetric in relation to the initial funding levels with both the 90% and 110% initial funds taking the same time to converge and similarly for the 80% and 120% initial funds. It is also clear that the difference in the years required for convergence between the 80% and 90% funds increases as the spread period increases.

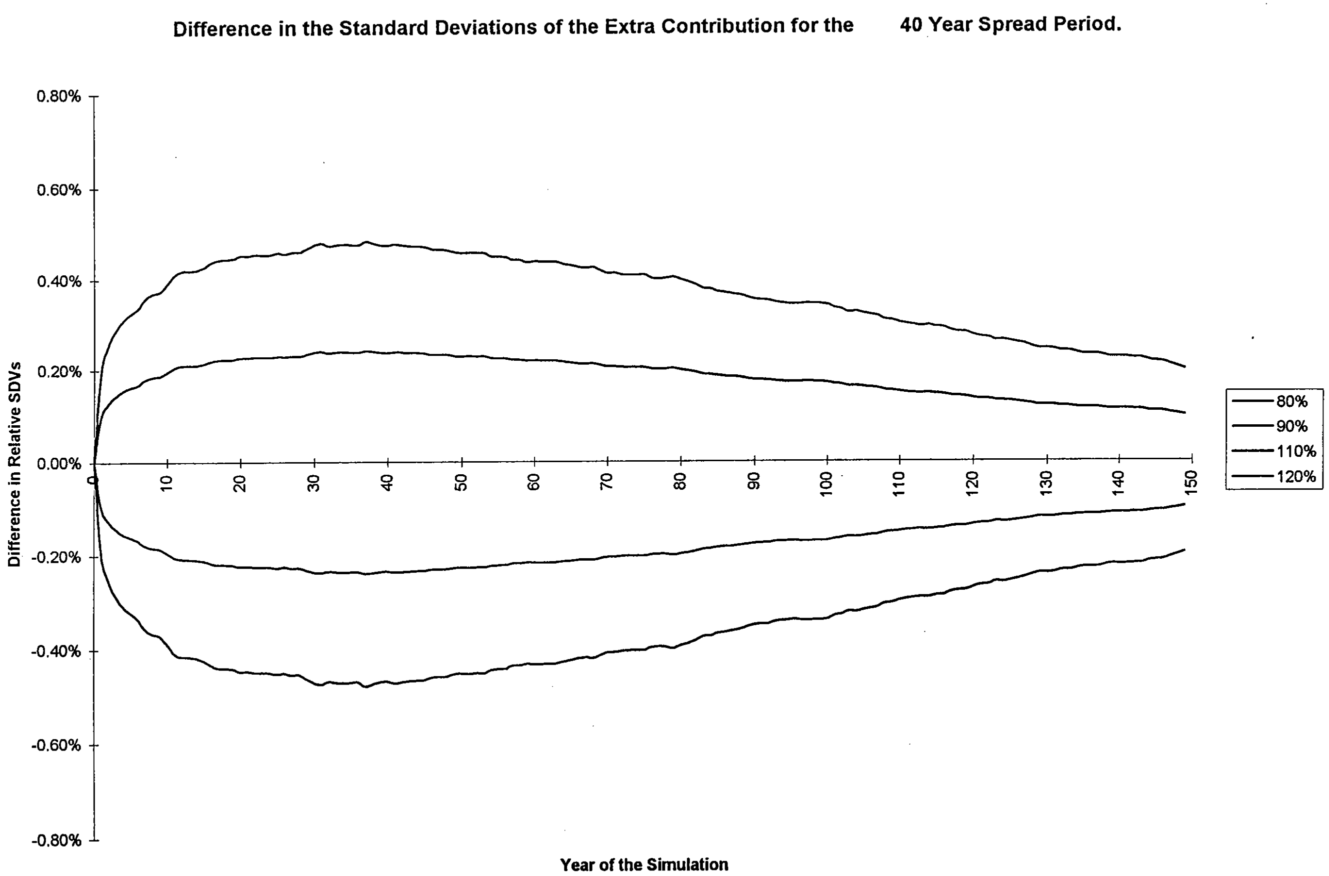
Figures 11.1 - 11.3 show the difference between the standard deviations of the extra contribution for the cases of the four different initial funding levels compared to that of the 100% initial fund for the spread periods 5, 20 and 40 years. These once again highlight the symmetry of the time to convergence and the differences in speed of convergence as the spread period is increased.

Difference in the Standard Deviations of the Extra Contribution for the 5 Year Spread Period.



Difference in the Standard Deviations of the Extra Contribution for the 20 Year Spread Period.




Difference in the Standard Deviations of the Extra Contribution for the 40 Year Spread Period.
Difference in Relative SDVs
0.80%
0.60%
0.40%
0.20%
0.00%
-0.20%
-0.40%
-0.60%
-0.80%
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
Year of the Simulation
80%
90%
110%
120%

12 Model IX - Different Spread Periods for Funding Surpluses and Deficits.

12.1 Description of Model IX.

For Model IX, the starting position of the model has been returned to the situation where the initial fund equals the initial actuarial liability. However, unlike the previous models so far tested Model IX uses two different spread periods. This is because, in practice, a funding deficit is likely to be perceived as being more adverse than a funding surplus and so Model IX uses a smaller spread period for amortizing deficits than the spread period used for amortizing surpluses (Winklevoss, 1993).

Model IX was simulated 2000 times for 149 years and with the pairs of spread periods of 5 and 10 years, 7 and 15 years, 10 and 20 years, 15 and 25 years, 20 and 30 years (the first spread period in each pair being the spread period used for amortizing deficits and the second in each pair being that used for amortizing surpluses).

12.2 Results.

The main aim in the analysis of the results from this model is to make a comparison with the results obtained from the single spread period model (Model VI). As the same attained annual rates of return from the investment model are being used for both Model VI and Model IX, the differences in the results for each year should be caused by the implementation of two spread periods rather than sampling errors.

To compare the effects of using two spreads on the standard deviation of the extra contribution, Table 12.2.1 shows the values for both Models VI and IX.

Table 12.2.1 - The Standard Deviations of the Extra Contribution and Fund for the Final Year of Simulation for Models VI and IX.

Spread Period (years)		Standard Deviation of the Extra Contribution in the Final Year (%)	
Model IX	Model VI	Model IX	Model VI
5	5/10	5.34	6.41
7	7/15	4.59	5.35
10	10/20	4.21	4.57
15	15/25	4.02	4.03
20	20/30	4.00	3.85

The results in Table 12.2.1 show that for the first four spread periods Model IX has a smaller standard deviation than Model VI. This is to be expected as the standard deviation of the extra contribution becomes smaller as the spread period is increased for the range of spread periods being considered. Therefore, for the 5/10 year combination spread period, it is reasonable to expect that the standard deviation of the extra contribution would lie between the standard deviation for the 5 year spread period and the standard deviation for the 10 year spread period and this is indeed the case. However, for the 20/30 year combination spread period, the standard deviation of the extra contribution is greater than for both the 20 year spread period and the 30 year spread period (3.92% from Table 9.2.1). In order to investigate why this has occurred, Tables 12.2.2 -12.2.4 show the distribution of the funds for the combinations of 5/10, 10/20 and 20/30 year spread periods and the corresponding single spread period fund distributions.

For each combination of spread period, the lower percentiles of the fund are compared to the lower percentiles of the single spread period whose length is the same as that used for funding a deficit in the combined spread period model. The reason for this is that the lower percentiles of the fund are where funding deficits occur so the combined spread period model (Model IX) would be using the shorter spread period. Similarly, the upper percentile of the fund is where over-funding is occurring and so the longer length of spread period would be used by the combined model. Therefore, the upper percentiles of the fund are compared to those of the single spread period model whose length corresponds to that used for funding surpluses in Model IX.

5/10 Year Spread Period Combination.

Table 12.2.2a - Fund Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread			Fund Percentile					
Period	Mean	Sdv	1%	5%	10%	25%	50%	IQR
5	23.07	2.17	18.73	19.84	20.37	21.48	22.96	3.04
5/10	23.85	2.63	19.10	20.13	20.70	21.95	23.59	3.56

Table 12.2.2b - Fund Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread			Fund Percentile					
Period	Mean	Sdv	50%	75%	90%	95%	99%	IQR
10	23.08	2.82	22.89	24.87	26.82	28.04	30.59	3.83
5/10	23.85	2.63	23.59	25.50	27.33	28.53	30.92	3.56

10/20 Year Spread Period Combination.

Table 12.2.3a - Fund Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Fund Percentile 1%	5%	10%	25%	50%	IQR
10	23.08	2.82	17.79	18.91	19.63	21.03	22.89	3.83
10/20	24.28	3.68	17.89	19.24	20.07	21.63	23.79	4.74

Table 12.2.3b - Fund Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Fund Percentile 50%	75%	90%	95%	99%	IQR
20	23.13	3.99	22.73	25.42	28.32	30.37	34.37	5.16
10/20	24.28	3.68	23.79	26.37	29.17	30.98	34.93	4.74

20/30 Year Spread Period Combination.

Table 12.2.4a - Fund Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Fund Percentile 1%	5%	10%	25%	50%	IQR
20	23.13	3.99	15.72	17.49	18.40	20.26	22.73	5.16
20/30	24.18	4.88	15.83	17.79	18.69	20.70	23.49	6.02

Table 12.2.4b - Fund Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Fund Percentile 50%	75%	90%	95%	99%	IQR
30	23.22	5.15	22.47	26.03	29.82	32.72	39.00	6.41
20/30	24.18	4.88	23.49	26.73	30.54	33.25	39.36	6.02

From Tables 12.2.2 - 12.2.4, it is clear that the combination spread period fund distributions have higher values for all their percentiles than the equivalent single spread period funds. This is an expected result when comparing the single spread period model (Model IV) that corresponds to the length of spread period used by the combined model for funding deficits: then, for funds that fall below the actuarial liability, the lengths of spread period used for both the single and the combined spread period model are identical and so both models are equally effective at returning the fund to the value of the actuarial liability. However, when the fund is greater than the value of the actuarial liability, the combined spread period model is less effective than the single spread period model at eliminating this over-funding. As the simulation progresses, the combined spread period model will, therefore, have more funds that are greater than the actuarial liability compared to the single spread period model and hence the value of the lower percentiles of the fund (i.e. those below 50%) will be greater than those for the single spread period model.

Similarly, when comparing the combined spread period model with the single spread period model corresponding to funding surpluses, it is expected that the combined spread period model will have more funds that are greater than the actuarial liability as it will be more effective at removing funding deficits than the single spread period model. Therefore, the value of the higher percentiles of the fund for the combined model will be greater than those for the single spread period model.

It can also be noted from Tables 12.2.2 - 12.2.4 that both the standard deviation and the inter-quartile range of the fund for the combined spread periods lie between the values for the two relevant single spread periods and also that the mean for the combined spread period case is greater than the mean for both the single spread period cases.

When we compare the differences between the percentiles of the fund for the single and combined spread periods that occur in Tables 12.2.4a and b to the differences that are recorded in Tables 12.2.2a and b and Tables 12.2.3a and b, there does not seem to be any clear reason why the standard deviation of the extra contribution for the 20/30 year spread period should behave differently to the standard deviations of the other combined spread periods. However, before investigating the actual distribution of the extra contribution, we consider two possible reasons for this behaviour.

Firstly, we consider the distribution of the funds for the combined spread periods. There does not appear to be the usual patterns for the mean and median i.e. the mean for the 10/20 year spread period is greater than that of the 5/10 year spread period which follows the pattern of Model VI (the mean value of the fund increases as the spread period increases), but the mean of the 20/30 year spread period lies between that of the 10/20 and 5/10 year spread periods. Similarly, instead of the median falling as the spread periods increase as was the case for Model VI, the median for the 10/20 year spread period is greater than that for the 5/10 year spread period whilst the median for the 20/30 year spread period is the smallest of the three. It would appear, therefore, that the introduction of two spread periods has affected the changes in the distributions of the fund as the spread periods are increased to an unexpected degree.

Secondly, we recall that when the model used the lower assumed rates of return (Model IV) and there was over-funding, the standard deviation of the extra contribution saw a rapid increase in its value when the spread period increased beyond 20 years (see Table 7.3.1). As the introduction of two spread periods has increased the amount of over-funding, this may be a possible explanation of why the standard deviation for the 20/30 year spread period is greater than the standard deviation of the 20 year spread period.

We next investigate why the combined spread period model for the 20/30 year spread periods has a larger standard deviation for the extra contribution than either of the corresponding single spread period models. Tables 12.2.5 and 12.2.6 show the distributions of the extra contribution for both the combined and single spread periods in terms of the percentiles for the combinations of 5/10 years and 20/30 years.

<u>5/10 Year Spread Period Combination.</u>

Table 12.2.5a - Extra Contribution Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Extra Contribution Percentile 1%	5%	10%	25%	50%	IQR
5	-0.24	6.41	-16.47	-11.13	-8.72	-4.49	0.22	8.90
5/10	-0.54	5.34	-13.03	-8.85	-6.94	-3.98	-0.95	7.11
10	-0.14	4.57	-12.50	-8.13	-6.23	-3.04	0.23	6.18

Table 12.2.5b - Extra Contribution Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Extra Contribution Percentile 50%	75%	90%	95%	99%	IQR
5	-0.24	6.41	0.22	4.41	7.80	9.26	12.10	8.90
5/10	-0.95	5.34	-0.95	3.13	6.87	8.50	11.61	7.11
10	-0.14	4.57	0.23	3.14	5.44	6.61	8.21	6.18

<u>20/30 Year Spread Period Combination.</u>

Table 12.2.6a - Extra Contribution Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Extra Contribution Percentile 1%	5%	10%	25%	50%	IQR
20	-0.13	3.85	-10.93	-7.11	-5.19	-2.34	0.24	4.97
20/30	-0.64	4.00	-12.47	-7.84	-5.64	-2.86	-0.37	5.06
30	-0.17	3.92	-12.09	-7.36	-5.18	-2.32	0.36	4.93

Table 12.2.6b - Extra Contribution Percentiles of the Final Year of the Simulation for Both Single and Combined Spread Period Models.

Spread Period	Mean	Sdv	Extra Contribution Percentile 50%	75%	90%	95%	99%	IQR
20	-0.13	3.85	0.24	2.63	4.48	5.42	6.90	4.97
20/30	-0.64	4.00	-0.37	2.20	4.10	5.15	6.79	5.06
30	-0.17	3.92	0.36	2.61	4.31	5.23	6.66	4.93

The relationship between the distribution of the extra contribution for the combined spread period and the distribution of the two relevant single spread periods is markedly different for the two combinations above. For the 5/10 year spread period (Tables

12.2.5a and b), all the percentiles of the combined spread period, with the exception of the 50% and 75%, lie between the percentiles of the two single spread periods while the 75% percentile lies only just outside (3.13% for the combined spread period compared to 3.14% for the 10 year single spread period). It is, therefore, straightforward to see why the value of the standard deviation for the combined spread period lies between the values for the two corresponding single spread periods.

In contrast, for the 20/30 year spread period, all the percentiles of the combined spread period are less than both the single spread period percentiles with the exception of the 99th percentile. From the distribution of the extra contribution for the combined spread period, it can be seen that the spread of the percentiles is greater than for both single spread periods. The reason for this is that the increase in over-funding caused by implementing the combined spread periods has decreased the value of the lower percentiles of the distribution of the extra contribution to a greater extent than the decrease in the higher percentiles. Therefore, the standard deviation of the extra contribution has increased when compared to both single spread periods.

The manner in which the shape of the distribution of the extra contribution changes when the combined spread period model is used appears to depend on the length of the different spread periods involved. This appears to be the reason why the changes in the distribution of both the fund and the extra contribution, as the spread periods are increased, do not correspond to the changes witnessed earlier when the spread period was increased in the case of Model VI.

13 Testing For Log-Normality in the Distribution of the Fund.

When describing the distribution of the fund for the different models in earlier sections, it has been mentioned that the distribution has approximately a log-normal shape. In this section, three of the models will be tested for log-normality - Model III, Model IV with 70% equities and Model VI with 70% equities - with the spread periods of 5, 20 and 40 years being considered.

The statistical data that were recorded for the funds were the mean, standard deviation and the percentiles of the fund. Using the mean and standard deviation, it is possible to calculate the percentiles of the fund if the fund were indeed log-normally distributed and in Tables 13.1 - 13.3 these theoretical values are compared to the values of the percentiles of the model.

Model III (LPI Pension Increases. 100% Equities)

5 Year Spread Period.

Table 13.1.a - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	16.76	18.33	19.23	20.82	22.75	24.86	26.92	28.24	30.89
Model	16.72	18.36	19.28	20.86	22.68	24.86	27.03	28.41	30.58
Difference	0.04	-0.03	-0.05	-0.04	0.07	0.00	-0.11	-0.17	0.31

20 Year Spread Period.

Table 13.1.b - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	13.60	16.07	17.56	20.37	24.02	28.33	32.87	35.92	42.44
Model	13.74	16.40	17.89	20.49	23.92	28.30	32.90	35.99	44.06
Difference	-0.14	-0.33	-0.33	-0.12	0.10	0.03	-0.03	-0.07	-1.62

40 Year Spread Period.

Table 13.1.c - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	10.03	13.23	15.34	19.64	25.85	34.01	43.55	50.49	66.62
Model	11.20	13.99	16.17	19.94	25.37	33.32	43.23	49.64	70.08
Difference	-1.17	-0.76	-0.83	-0.30	0.48	0.69	0.32	0.85	-3.45

From the differences between the actual values of the fund percentiles and the theoretical values presented in Table 13.1a, it can be seen that the log-normal distribution could be used to approximate the fund distribution for Model III when the spread period is 5 years, as none of the differences is particularly large and their signs are neither consistently positive or negative.

The use of a log-normal distribution to approximate Model III when the spread period is increased to 20 years becomes more tenuous as Table 13.1b shows that the differences between the actual percentiles and the theoretical percentiles have generally increased. Also, there are systematic departures and only the 50% and 75% percentiles now have a positive difference with all the other percentiles having negative differences.

When the spread period for Model III is increased to 40 years, the differences between the theoretical and actual percentile values increase again although there are now four positive differences and five negative differences. However, as all the positive differences are in a cluster rather than spread throughout the percentiles, it would certainly appear that a better approximation to the fund distribution than the log-normal distribution would be needed for practical applications.

Model IV (70% Equities, 30% ILGs - Original Parameters)

5 Year Spread Period.

Table 13.2.a - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	19.33	20.62	21.34	22.60	24.09	25.68	27.20	28.15	30.03
Model	19.47	20.75	21.39	22.60	24.07	25.56	27.24	28.17	30.17
Difference	-0.14	-0.13	-0.05	0.00	0.02	0.12	-0.04	-0.02	-0.14

20 Year Spread Period.

Table 13.2.b - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	17.08	19.24	20.50	22.79	25.64	28.85	32.08	34.19	38.51
Model	17.29	19.38	20.62	22.82	25.56	28.73	32.01	34.15	39.12
Difference	-0.21	-0.14	-0.12	-0.03	0.08	0.12	0.07	0.04	-0.61

40 Year Spread Period.

Table 13.2.c - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	14.85	17.97	19.89	23.56	28.44	34.34	40.68	45.03	54.46
Model	15.95	18.37	20.06	23.62	28.52	33.88	40.43	44.95	55.65
Difference	-1.10	-0.40	-0.17	-0.06	-0.08	0.46	0.25	0.08	-1.19

From Table 13.2a, it can be seen that using a log-normal approximation could be justified for modelling the fund distribution for Model IV when the spread period is 5 years, as the difference between the theoretical and observed percentile values are relatively small. However, the fact that the differences are only positive for the 25%, 50% and 75% percentile and that the negative differences are greatest for the out-lying percentiles would indicate that the fund distribution is not exactly log-normal.

The pattern of differences in Table 13.2b is very similar to that of Table 13.1c with the 50%, 75%, 90% and 95% percentiles being positive and the others being negative. This would indicate that the distribution of the fund is not log-normal for the 20 year spread period.

Increasing the spread period to forty years for Model IV makes the distribution of the fund appear to be even less log-normally distributed than for the twenty year spread period, as the differences between the theoretical values and the observed values have increased and also the difference for the median has become negative.

Model VI (70% Equities, 30% ILGs - Altered Parameters)

5 Year Spread Period.

Table 13.3.a - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	18.46	19.68	20.36	21.56	22.97	24.48	25.91	26.81	28.59
Model	18.73	19.84	20.37	21.48	22.96	24.52	25.91	26.75	28.72
Difference	-0.27	-0.16	-0.01	0.08	0.01	-0.04	0.00	0.06	-0.13

20 Year Spread Period.

Table 13.3.b - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	15.31	17.20	18.30	20.31	22.80	25.59	28.39	30.21	33.95
Model	15.72	17.49	18.40	20.26	22.73	25.42	28.32	30.37	34.37
Difference	-0.41	-0.29	-0.10	0.05	0.07	0.17	0.07	-0.16	-0.42

40 Year Spread Period.

Table 13.3.c - Percentiles of the Theoretical and Simulated Final Year Fund Distributions.

	Percentile								
	1%	5%	10%	25%	50%	75%	90%	95%	99%
Theoretical	12.14	14.55	16.02	18.81	22.50	26.90	31.59	34.79	41.67
Model	12.83	14.91	16.36	18.85	22.27	26.63	31.49	34.79	43.14
Difference	-0.69	-0.36	-0.34	-0.04	0.23	0.27	0.10	0.00	-1.47

From Table 13.3a, it appears that the fund for the five year spread period for Model VI could be approximated by the log-normal distribution as all the recorded percentiles between the 10th and 95th are similar to the calculated percentiles. However, the differences between the observed and calculated values for the 1%, 5% and 99% percentiles would indicate that the log-normal distribution would need to be improved for particular practical applications.

Similarly, as the spread period is increased to twenty years (Table 13.3b) and forty years (Table 13.3c), the pattern of the log-normal approximation becoming less appropriate is repeated as the differences between the observed percentiles and the calculated percentiles increase.

In conclusion, it appears that the distribution of the fund is not actually log-normally distributed but this approximation could be used for the smaller spread periods. Because the data for each simulation were not recorded, it is not possible to test the distribution of the fund against other skewed distributions such as the inverse gamma distribution: there are theoretical reasons for suggesting this distribution - Dufresne (1990) shows that the present value of a perpetuity, when the force of interest follows a Brownian motion process (the continuous time analogue of the IID assumption), has, in the limit, an inverse gamma distribution.[1]

[1] If X has an Inverse Gamma distribution with parameters α and λ, then X^{-1} is distributed as Gamma (α,λ) and hence it can be shown that $E(x) = \dfrac{\lambda}{\alpha - 1}$, $Var(x) = \dfrac{\lambda^2}{(\alpha-1)^2(\alpha-2)}$ and the skewness coefficient is $\dfrac{4(\alpha-2)^{1/2}}{\alpha-3}$ for $\alpha>3$.

14 Conclusions.

The aim of this project was to test if the theoretical results obtained from the IID, AR(1) and MA(1) models about the behaviour of the optimal spread period would be supported by the results obtained from a more realistic model of investment returns. In particular, we wanted to see if the results held as the model was developed, becoming more realistic and inevitably gaining in complexity. The model used has included the Wilkie model of stochastic investment returns, the selection of assets from three categories (equities, ILGs, Consols), and benefits uprated in line with inflation (salary inflation, then price inflation and then LPI). We have seen that each model has given the expected result on the change in the length of the optimal spread period (when compared to the previous model) as the mean and standard deviation of the pension scheme's investment portfolio has changed.

The results obtained from Model VII (section 10) have also supported the theoretical conclusion (Haberman 1993) that moving from annual to triennial valuations would increase the standard deviation of both the fund and the extra contribution and have a minor effect on the range of the optimal spread periods.

The results from Model VIII (section 11) have supported the theoretical conclusions about how the standard deviation of the extra contribution would be affected in both the long and the short term by different initial funding levels.

As well as supporting the theoretical results this project has also identified other areas of investigation. Firstly, the change in the length of the optimal spread period when the model was changed from Model IV to Model VI (where the assumed rates of return were altered) would appear to indicate that the length of the optimal spread period is determined by the assumed rates of return used to calculate the contribution rate and the liabilities as well as the actual underlying rates of return. There would therefore appear to be an improvement in the model if the rates of return used to calculate the contributions and liabilities were altered during the simulation to take into account past experience.

Secondly, the introduction of a second spread period into the model (Model IX - section 12) so that a different spread period is used for amortizing surpluses and deficits appears to alter the distributions of both the fund and the extra contribution and also appears to change the range of the optimal spread periods.

Finally, when the selected models were tested to see if the fund was log-normally distributed (section 13), the hypothesis of log-normality was rejected. But this leaves the possibility that the fund distribution could be approximated by another skewed distribution and the theoretical suggestion of an inverse gamma distribution will be left to future research.

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Appendix.

The following appendix is a description of the background theoretical results and is based on sections of Haberman (1994b). It relates to individual funding methods and the spread method for adjusting contributions.

C(t) = Contribution Rate in year t.

F(t) = Fund at time t.

NC(t) = Normal Cost in year t.

AL(t) = Actuarial Liability at time t.

ADJ(t) = Adjustment to the Contribution Rate at time t.

UL(t) = Unfunded Liability at time t.

A1 THE MATHEMATICAL MODEL

At any discrete time t (for integer values t = 0, 1, 2 ...), a valuation is carried out to estimate C(t) and F(t), based only on the scheme membership at time t. However, as t changes, we allow for new entrants to the membership so that the population remains stationary - see assumptions below.

In the mathematical discussion, we make the following assumptions.

1. All actuarial assumptions are consistently borne out by experience, **except** for investment returns.

2. The population is stationary from the start. (We could alternatively assume that the population is growing at a fixed, deterministic rate i.e. that the population is **stable** in the sense of Keyfitz (1985)).

3. Inflation on salaries at a deterministic rate is incorporated by considering interest rates that are *"real"* relative to salaries. In parallel we assume that benefits in payment increase at the same rate as salaries. We therefore consider variables to be in real terms. For simplicity, each active member's annual salary is set at 1 unit at entry. (It would be possible to incorporate a fixed promotional salary scale simply through a change of notation).

4. The interest rate assumption for valuation purposes is fixed, i_v.

5. The *"real"* interest rate earned on the fund during the period, $(t, t + 1)$ is $i(t + 1)$. The corresponding *"real"* force of interest is assumed here to be constant over the interval $(t,t+1)$ and is written as $\delta(t + 1)$. Thus, $1 + i(t + 1) = \exp(\delta(t + 1))$. $i(t)$ is defined in a manner consistent with the definition of $F(t)$.

6. We define $E[1+i(t)] = E[\exp \delta(t)] = 1+i$. We assume that $i=i_v$ where i_v is the valuation rate of interest. This means that the valuation rate is correct ***"on average"***. This assumption is not essential mathematically but it is in agreement with classical ideas on pension fund valuation.

7. It is assumed that the contribution income and benefit outgo occur at the start of each period (or scheme year).

8. The initial value of the fund (at time zero) is known, i.e. Prob $[F(0)=F_0]=1$ for some F_0.

9. Valuations are carried out at annual intervals (this is relaxed in section A4).

Assumptions 1, 2, 3 and 4, imply that the following parameters are constant with respect to time, t (after rescaling to allow for growth in line with salary inflation):

NC the total normal contribution

AL the total actuarial liability

B the overall benefit outgo (per unit of time)

S the total pensionable payroll

PVB the present value of future benefits (for active members and pensioners)

PVS the present value of future pensionable earnings.

Further, assumptions 1, 2, 4, 7 and 9 imply that the following equation of equilibrium holds:

$AL = (1+i)\,(AL + NC - B)$ or equivalently $B = d.AL + NC$ (A1)

where $d = i(1 + i)^{-1}$, the compound interest discount rate.

This equation of equilibrium can also be found in the earlier papers of Trowbridge (1952) and Bowers et al (1976).

The model adopts a discrete time (rather than continuous time) approach.

Under the spread method, ADJ(t) = k. UL(t) (A2)

where $k=(\ddot{a}_{\overline{M|}})^{-1}$ calculated at the valuation rate of interest. So the unfunded liability is spread over M years, where M would be chosen by the actuary. It should be noted that this definition of ADJ(t) uses the same fraction of the unfunded liability regardless of the sign of the latter. So, surpluses and deficiencies would be treated in a comparable manner - this would not always be the case in practice: different choices of M depending on whether there is a surplus or deficit have been considered in section 12. k is the fraction of UL(t) that makes up ADJ(t) and can be thought of as a penalty rate of interest that is being charged on the unfunded liability, UL(t).

Then F(t) = AL(t) - UL(t) and C(t) = NC(t) + ADJ(t).

A2 STOCHASTIC INVESTMENT RETURNS

Independent and Identically Distributed i(t)

We assume that the earned real rates of investment return, i(t) for $t \geq 1$, are independent and identically distributed random variables, with $i(t) > -1$ with probability 1, and with $Ei(t) = i = i_v$ and $\mathrm{Var}\, i(t) = \sigma^2 < \infty$.

Dufresne (1988) has described in detail the properties of individual funding methods.

As we are using the spread method,

$$C(t) = NC + k\,(AL - F(t)) \qquad \text{(A3)}$$

and

$$F(t+1) = (1+i(t+1))(F(t)+C(t)-B) \qquad \text{(A4)}$$

Equation (3) includes a negative feedback component, whereby the current status, F(t), is compared with a target (AL) and corrective action is taken to deal with any discrepancy.

Then, Dufresne (1988) shows that

$$E\,F(t) = q^t F_0 + r(1-q^t)/(1-q) \qquad \text{(A5)}$$

where $q = (1+ i)(1 - k)$ and $r = (1 + i)(NC + k.AL-B)$,

and
$$\text{Var F(t)} = b \sum_{j=1}^{t} a^{t-j} \, (E\,F(j))^2 \qquad \text{(A6)}$$

where $a = q^2 (1 + b)$ and $b = \sigma^2(1 + I)^{-2}$.

Then, $q=\dfrac{\ddot{a}_{\overline{M-1}|}}{\ddot{a}_{\overline{M}|}}$ so if M >1, 0 < q < 1 and the following limits exist

and

$$\left.\begin{array}{l} \lim\limits_{t \to \infty} E\,F(t) = \dfrac{r}{1-q} = AL, \text{ using (3)} \\ \text{and} \\ \lim\limits_{t \to \infty} E\,C(t) = NC, \text{ using (7)} \end{array}\right\} \qquad \text{(A7)}$$

If a < 1, then Dufresne (1988) shows that

$$\left.\begin{array}{l} \lim\limits_{t \to \infty} \text{Var F(t)} = \dfrac{\text{bAL}^2}{(1-\text{a})} \\ \text{and} \\ \lim\limits_{t \to \infty} \text{Var C(t)} = \dfrac{\text{bk}^2\text{AL}^2}{(1-\text{a})} \end{array}\right\} \qquad \text{(A8)}$$

If a$\geq$ 1, then both of these limiting variances would be infinite. The restriction that a < 1 implicitly places a restriction on the choice of M viz

$$\text{a} < 1 \text{ is equivalent to } \ddot{a}_{\overline{M}|} < \frac{1}{1-f} \text{ where } f = v\sqrt{\frac{1}{1-b}}.$$

This is equivalent to $M < M_0 = \frac{1}{\delta}\ln\left[\frac{(1+i)(1+b)^{1/2}-1}{(1+b)^{1/2}-1}\right]$ and provides a restriction on the feasible values of M for convergence. Numerical values of M_0 have been given in Table 2.4 (see section 2).

Dufresne (1988) also considers expressions for the covariances of F and C in the limit and deals separately with the special case M = 1.

A3 OPTIMAL SPREAD PERIODS.

In this section we focus on individual funding methods with the spread method, and we shall consider the existence of an ***"optimal"*** region for the spread period.

In this section, we shall consider the relationship between Var F(t) and Var C(t) as M (or k) varies, with t fixed. Rather than take a particular finite t, we shall consider the limiting variances at $t \to \infty$ and indeed we shall consider these variances relative to the corresponding expectations (i.e. the coefficient of variation). Our consideration of the case where $t \to \infty$ is justified on the grounds that the results are mathematically tractable. The earlier discussion, leading to equation (A8), shows that the limiting variances are both proportional to AL^2 so that a more secure funding method, with higher AL, would lead to greater variability. The converse would suggest that the limiting variances could be reduced, in absolute terms, by choosing a funding method with a lower value of AL. This difficulty is by-passed by our focusing on the coefficients of variation. To facilitate the discussion, we now introduce some new notation.

With a < 1 and $2 \leq M < \infty$ (so d < k <1), we define

$$\alpha(k) = \frac{VarF(\infty)}{(EF(\infty))^2} \text{ and } \beta(k) = \frac{VarC(\infty)}{(EC(\infty))^2} \tag{A9}$$

and we regard α and β as functions of k. We could equivalently regard them as functions of M, given the 1-1 correspondence between k and M. However, it is more convenient to consider $\alpha(k)$ and $\beta(k)$.

For the case of IID i(t), Dufresne (1988) has considered in detail the trade off between Var F(t) and Var C(t) in the limit as $t \rightarrow \infty$, as represented by $\alpha(k)$ and $\beta(k)$, and for finite t under certain conditions. Thus, from (A7) and (A8), we have that

$$\alpha(k) = \frac{b}{1-y(1-k)^2}$$

$$\text{and } \beta(k) = \frac{AL^2}{NC^2} \cdot \frac{bk^2}{1-y(1-k)^2}$$

where $y=(1+i)^2(1+b) = E(1+i(t))^2$. Assuming that y > 1, Dufresne shows that

$$\frac{d}{dk}\alpha(k) < 0$$

$$\text{and} \frac{d}{dk}\beta(k)\Big|_{k^*} = 0 \text{ where } k^* = 1-\frac{1}{y}.$$

At $k = k^*$, $\beta(k)$ takes a minimum value. The value of the spread period corresponding to k^* will be denoted by M^*.

Formally, if y > 1, then both Var F(∞) and Var C(∞) become infinite for some finite M = M_0 (when a becomes equal to 1) and there exists a value M^* such that

- for $M \le M^*$, Var F(∞) increases and Var C(∞) decreases with M increasing
- for $M \ge M^*$ both Var F(∞) and Var C(∞) increase with M increasing.

If y = 1, Var C(∞) → 0 and Var F(∞) → ∞ as M → ∞, although Var F(∞) does stay finite for all M.

If y < 1, Var C(∞) → 0 as M → ∞ and Var F(∞) has a finite limit as M → ∞.

The particular value of M^* is determined by

$$k^* = 1 - \frac{1}{y} \quad = \quad \frac{1}{\ddot{a}_{\overline{M|}}}$$

$$\text{i.e. if } i \neq 0 \; M^* = -\frac{1}{\delta} \ln\left(\frac{vy-1}{y-1}\right) \qquad \text{(A10)}$$

$$\text{and if } i = 0 \; M^* = 1 + \frac{1}{\sigma^2}.$$

There is thus a trade off between variability in the fund, represented by α, and variability in the contribution rate, represented by β. This trade-off takes place only up to $M = M^*$. Beyond this point, augmenting M causes both Var F and Var C to increase, With the objective of minimising variances, any choice of $M > M^*$ should be rejected, for clearly some $M < M^*$ would reduce both Var F and Var C. If we regard M as being a parameter open to the choice of the actuary, then the optimal choices for M would lie in the region $1 \leq M \leq M^*$. Thus, we can describe this region as an ***"optimal"*** region and M* as the 'critical' spread period.

Table 2.1 in section 2 provides values of M^* as a function of i and σ (to the nearest integer). In the UK, it is common to choose M to correspond to the average remaining working lifetime of the current membership - with an average age of membership of 40-45 and a normal retirement age of 65 this would correspond to a choice of M in the range 20-25. We see from Table 2.1, that under particular combinations of i and σ our model indicates that this choice would **not** be optimal. If $i = .03$ and $\sigma = .20$ then, for example, smaller values, namely those in the region $1 \leq M \leq 13$, would be more satisfactory.

The optimal range of spread periods has also been demonstrated to exist in the case of finite t, for t 'large enough': further details are given in Owadally and Haberman (1995).

A4 FREQUENCY OF VALUATIONS

A second control variable available to the actuary is the frequency with which valuations are performed. In the model in sections 1 - 3, we have assumed that valuations are annual. Here, we shall consider the case of valuations every 3 years (and then more generally every n years where n is an integer). Triennial valuations are common in the UK because of legislative and cost considerations. We here introduce j(t) to be the real rate of investment return earned during the t'th (three year) period.

In the triennial case, the equation of equilibrium would become

$$AL = (1 + j)(AL + NC' - B') \tag{A11}$$

where NC′ and B′ now refer to 3 year rather than 1 year time periods and $(1 + j) = (1 + i)^3$.

The link between the pairs NC and NC′, B and B′ comes from the following straightforward compound interest relationships:

$$NC' = NC\ \ddot{a}_{\overline{3}|} \text{ and } B' = B\ \ddot{a}_{\overline{3}|}. \tag{A12}$$

Now equation (A3) would become

$$C(t) = NC' + k_1\ (AL - F(t)), \text{ for } t=0,1,2, \ldots \tag{A13}$$

where $k_1 = 1/\ddot{a}^{j}_{\overline{M/3}|}$ calculated at the real rate of interest j effective over a triennium and corresponding to i effective per year. We note that in equation (13), t is measured in 3 year time units (rather than annual units as in equation (3)).

As noted earlier, $1 + j = (1 + i)^3$

and so $$k_1 = \frac{1 - v^3}{1 - v^M} \text{ where } v = (1 + i)^{-1}.$$

We assume here that the contributions are paid at the start of each triennium. In reality, they would be paid annually; however, this feature introduces complexity into the mathematical formulation. By effectively working in 3 year time units, we avoid such complications.

Haberman (1993) derives equations that correspond directly with (A7) and (A8). Thus, if M>3,

$$\lim_{t\to\infty} E\, F(t) = AL \quad \text{and} \quad \lim_{t\to\infty} E\, C(t) = NC' \tag{A14}$$

And, providing that $y_1(1 - k_1)^2 < 1$ where $y_1 = E\,(1+j(t))^2 = (1 + 2i + i^2 + \sigma^2)^3 = y^3$,

$$\left.\begin{aligned} \lim_{t\to\infty} Var\, F(t) &= \frac{(Var j(t))\, AL^2}{(1+j)^2\left(1 - y_1(1-k_1)^2\right)} \\ \lim_{t\to\infty} Var\, C(t) &= \frac{(Var j(t))\, k_1^2\, AL^2}{(1+j)^2\left(1 - y_1(1-k_1)^2\right)} \end{aligned}\right\} \tag{A15}$$

We now extend the definitions of α and β in equations (9) so that $\alpha_0(k)$ and $\beta_0(k)$ refer to the annual case and $\alpha_1(k_1)$ and $\beta_1(k_1)$ refer to the triennial case. Then, Haberman (1993) demonstrates that an optimal spread period, M_1^*, exists for the triennial case providing that $y_1>1$. The corresponding value of k_1 is

$$k_1^* = 1 - \frac{1}{y_1} = 1 - \frac{1}{y^3} \text{ (for } i \neq 0).$$

Comparison of the resulting values of M^* (annual case) and M_1^* (triennial case) indicates that

$$M_1^* \approx M^* + 1.$$

Haberman (1993) also compares the limiting variances in the annual and triennial cases (equations (8) and (15)) and obtains ranges for the spread period for which the variances are increased in the triennial case relative to the annual case. The existence of a spread period M_3 is demonstrated for which, in the triennial case, the relative limiting variances of both F(t) and C(t) are increased for values of M in the range $(1, M_3)$. M_3 and M_1^* are found to be approximately equal, i.e. $M_3 \approx$

$M_1^* \approx M^* + 1$. This leads to the intuitively reasonable result that, with triennial valuations and a sensible choice of the spread period (i.e. within the optimal range), the limiting variances of <u>both</u> F(t) and C(t) are increased relative to the case where valuations are annual.

Haberman (1993) also demonstrates how these results may be generalized to apply to valuations avery n years, where n is an integer. Similar results to (14) and (15) can be derived, for example, but expressed in terms of $k_n = \frac{1-v^n}{1-v^M}$, say, $(1+i)^{2n}$ and y^n. Again, the existence of a range of spread periods is considered for which, relative to the annual case, the relative limiting variances of <u>both</u> F(t) and C(t) are increased.

A5. INTRODUCTION TO AR(1) AND MA(1) INVESTMENT RETURN MODELS.

A5.1 First Order Autoregressive Model.

Haberman (1994a) has extended the investment return model to a first order auto-regressive model. Under this model the (earned real) force of interest is given by the following stationary (unconditional) autoregressive process in discrete time:

$$\delta(t) = \theta + \varphi[\delta(t-1) - \theta] + e(t)$$

where e(t) for t = 1, 2, ... are independent and identically distributed normal random variables each with mean 0 and variance γ^2.

The model suggests that interest rates earned in any year depend upon interest rates earned in the previous year and some constant level.

The condition for this process to be stationary is that $|\varphi| < 1$.

The relative standard deviations for the fund and contribution rate for certain parameter values and for different lengths of spread periods is given in Table 2.2 (section 2).

A5.2 First Order Moving Average Model.

Haberman and Wong (1997) have suggested the following first order stationary (unconditional) moving average process in discrete time for the (earned real) force of interest:

$$\delta(t) = \theta + e(t) - \varphi e(t-1)$$

where e(t) for t = 1, 2, ... are independent and identically distributed normal random variables each with mean 0 and variance γ^2. φ is the moving average parameter.

A moving average process is second-order stationary regardless of the value of φ. The invertability condition for this model is $-1 < \varphi < 1$.

The relative standard deviations for the fund and contribution rate for certain parameter values and for different lengths of spread periods is given in Table 2.3 (section 2).

A5.2 <u>Range of Optimal Spread Periods.</u>

As noted in section 2, there is evidence reported in Haberman (1994a) and Haberman and Wong (1997) that a range of optimal spread periods exists in the presence respectively of AR(1) and MA(1) investment returns, providing certain constraints on the investment return parameters are satisfied. For more details, interested readers are referred to the original papers. Also, as in the IID case, the range of optimal spread periods decreases as the standard deviation of the underlying investment return process increases.